1917
H864

Huffer

Continuity of a Derivative Defined by the Law of the Mean

CONTINUITY OF A DERIVATIVE DEFINED BY THE LAW OF THE MEAN

BY

CHARLES MORSE HUFFER

A. B. Albion College, 1916

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF ARTS

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1917

1917
H864

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

May 28 1917

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Charles Morse Huffer

ENTITLED CONTINUITY OF A DERIVATIVE DEFINED BY THE LAW OF THE MEAN

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Master of Arts

In Charge of Thesis

Head of Department

Recommendation concurred in:*

Committee on Final Examination*

*Required for doctor's degree but not for master's.

6 Feb. 17 Gilley

376.15

UIUC

INDEX.

CHAPTER I.

Functions of One Variable.

Art.		Page.
1.	History of the Law of the Mean and of Rolle's Theorem.	1
2.	Statements of the Law of the Mean.	3
3.	Proof of the Law of the Mean.	4
4.	Examples showing the conditions of the Law of the Mean not necessary.	6
5.	Extensions of the Law of the Mean.	9
6.	Outline of Paper by E.R.Hedrick, Annals of Mathematics, July, 1906.	13
7.	Added Problems and Investigations.	20
8.	A condition for the continuity of $\xi(h)$.	27

CHAPTER II.

Functions of Two Variables.

9.	Statements of the Forms of the Theorem.	30
10.	Proofs.	31
11.	Examples showing the conditions sufficient but not necessary.	35
12.	Geometric Interpretation.	36
13.	Hedrick's theorems extended to functions of two variables.	40

PREFACE.

The object of this thesis is to study the Law of the Mean and its relation to derivatives, not only for functions of one variable but extended to functions of two variables. The work is divided into two chapters; the first chapter being devoted to the study of functions of a single variable, and the second to functions of two variables.

In the first chapter, the statement and proof of the Law of the Mean is given with examples showing the conditions given in the statement of the theorem to be sufficient but not necessary. An example is given of a function whose derivative does not exist at the rational points but the Law of the Mean holds for any interval. Section five is given to some extensions of the Law of the Mean as given by Dini in his book on Function Theory. The two following sections contain a review of an article written by Professor E.R. Hedrick, published in the Annals in July 1906, and some added problems following the same line. This article studies the relation of the function $\theta(x,h)$ defined by the Law of the Mean to the continuity of the derivative $f'(x)$. It is found that the continuity of θ is a sufficient but not necessary condition for the continuity of the derivative.

In Chapter II the Law of the Mean for functions of two variables is proved in three different forms. The second form is the most important and forms the basis for the study of the remainder of the chapter. The conditions given in these three forms are shown to be sufficient but not necessary. One section of this chapter is given over to a study of the differentiability of $f(x,y)$ under the conditions given in the three forms of the theorem and to a geometrical interpretation of the theorem in the second form. The chapter closes with the extension of the work given in Hedrick's article in the Annals to functions of two variables.

Digitized by the Internet Archive
in 2013

http://archive.org/details/continuityofderi00huff

CHAPTER I.

THE LAW OF THE MEAN FOR FUNCTIONS OF ONE VARIABLE.

1. HISTORY OF THE LAW OF THE MEAN AND OF ROLLE'S THEOREM.

The first statement of the theorem of the mean was given by Cavalieri in the year 1635, the theorem appearing in its geometric form; that is, the parallelism of a chord cutting a given curve, and the tangent at a point of the curve between the points of intersection of the chord with the curve.

The statement as given by Cavalieri is as follows: (1)
Si curva linea quaecunque data tota sit in eodem plano, cui occurrat recta in duobus punctis..., poterimus aliam rectam lineam prefatae aequidistantem ducere, quae tangat portionem curvae lineae inter duos praedictos occursus continuatam. (2)

Cavalieri's statement of the theorem may be translated thus:
If any curved line is given lying wholly in a plane, and is cut in two points by a straight line..., we con draw another straight line equally distant from the first straight line, and which touches the continuous portion of the curve between the two points in which the curve is cut by the secant line.

Rolle's theorem appeared for the first time in 1690 in his book *"Methode pour Rèsoudre les égalitéz, etc."* This book is to be found in only three libraries in France; the *"Bibliothèque Nationale"*, the *"Bibliothèque de l'Arsenal"*, and the *"Bibliothèque de L'Institute de France"* in Paris. (3)

Rolle states the theorem which bears his name as follows:
Lorsqu'il y a racines effective dans une cascade, les hypothèses de cette cascade donnent alternativement l'une + et l'autre -.

Rolle's statement of the theorem may be translated:
When there are real roots in an equation, the roots of the first derivative of this equation give alternately one root positive and the following root negative.

---

(1) Cavalieri, *Geometria Indivisib. etc., 1635*; p. 492 edition 1653.
(2) G. Vacca, *Bibl. Math., 1901.* p. 149.
(3) Cajori, *Bibl. Math., 1910-11.* p. 300.

Another statement of Rolle's Theorem has been given which states clearly Rolle's meaning in terms understood by the modern reader: [1] *If an algebraic equation $f(x)=0$ has only real and unequal roots, and if a and b are two consecutive roots of the equation $f'(x)=0$, then one of the roots of the equation $f(x)=0$ is located between a and b.*

Cantor [2] states the theorem given by Rolle: *Zwischen zwei aufeinander folgenden Wurzeln α und β der Gleichung $f'(z)=0$, kann nicht mehr als eine einzige Wurzel von $f(z)=0$ liegen.*

Charles Reyneau stated Rolle's Theorem in connection with equations free from complex roots in his *Analyse Demontree* in 1708. L.Euler in 1755 made statements which mean practically the same as the theorem given by Rolle. The theorem was applied to equations having complex roots by J.R.Mourraille in 1768. Lagrange gave the theorem more prominence in his study of algebraic equations. [3]

These earlier writers and the other writers of that time give the theorem first given by Rolle, but do not attribute it to him. It was not until 1834 that any writer gave Rolle the credit for the theorem which now bears his name. In that year M.W.Drobisch gave the theorem and called it Rolle's Theorem. [4] He also gave a corollary which he called Rolle's Corollary. Since that time, writers who have given the theorem have called it Rolle's Theorem.

At first, as we have seen, the Law of the Mean and Rolle's Theorem existed apart. The former was given by Lagrange in 1797 as a consequence of Taylor's Theorem. [5] It was given by Ampere and then by Cauchy. [4] Ossian Bonnet [6] brought the Law of the Mean and Rolle's Theorem together, giving the first proof of the Law of the Mean by means of Rolle's Theorem in practically the same form as it is given at the present time.

---

(1) *Ency. des Sc. Math.*, and *Bibl. Math 1906-7.* p. 301; *1900.* p. 454; *1903*
(2) Cantor, *Vorlesung.* Vol. III, p. 128.
Note. Mention is made of the history of Rolle's Theorem by the historians Marie and Montucla.
(3) Lagrange. *Resolution des équations numeriques de tous degres. 1798*
(4) Cajori. *Bibl. Math 1910-11.* p. 300.
(5) Lagrange. *Oevres IX Paris 1881.* p. 83
(6) Serret. *Cours de calcul diff. et int. 1868.* p. 17, Alsp Ed. 1876.

2. STATEMENTS OF THE LAW OF THE MEAN.

The Theorem of the Mean has been stated in several ways by different writers. The meaning of the theorem in every case however means practically the same. The conditions given have been shown by Hedrick [1] to be sufficient but not necessary. I give here the statements of the theorem as given by six recent writers.

1. Pierpont. [2] Let $f(x)$ be continuous in $A = (a,b)$ and let $f'(x)$ be finite or infinite within A. Then for some point $a < c < b$,

$$f(b) - f(a) = (b-a)\cdot f'(c).$$

2. Neuberg. [3] If a function $F(x)$, finite and continuous in the interval (a,b), admits a unique and definite derivative for each value of x taken between a and b, there exists a number c lying between a and b, such that we have

$$F(b) - F(a) = (b-a)\cdot F'(c).$$

3. Pascal. [4] If $f(x)$ possesses a derivative for all values of x in the interval (a,b), then there is at least one value ξ of the variable within the interval, such that

$$f(b) - f(a) = (b-a)\cdot f'(\xi).$$

4. Lamb. [5] If a function $\varphi(x)$ be continuous and have a derivative determinate throughout the interval from $x = a$ to $x = b$, then for x_1 within the interval

$$\varphi(b) - \varphi(a) = (b-a)\cdot \varphi'(x_1).$$

5. Hobson. [6] If a function $f(x)$ be continuous in the interval $(x,x+h)$ and at every point in the interior of this interval $f'(x)$ exist, being either finite or infinite with fixed sign, then a point $x+\theta h$ exists, where θ is some proper fraction and is neither 0 nor 1 such that

$$f(x+h) = f(x) + h\cdot f'(x+\theta h).$$

6. Encyclopédie des Science Mathematiques. [7] If $f(x)$ be a function defined in the interval (a,b), being finite and continuous in x in the interval; and admitting at the interior of

---

(1) *Annals of Mathematics.* July 1906. p.181.
(2) *Theory of Functions of Real Variables.* p.248.
(3) *Cours d'Analyse Infinitesimale.* p.50.
(4) Repertorium. (1910) p.469.
(5) *Infinitesimal Calculus.* p.129.
(6) *Theory of Functions of a Real Variable.* Par.203.
(7) Tome II, Vol.1, Fasc.2. p.270.

(a,b) a definite derivative which is in general finite but may possibly be $+\infty$ or $-\infty$ in some isolated points of (a,b); then we have for at least one value of the real number θ, taken between 0 and 1,

$$f(b) - f(a) = (b-a)\cdot f'(\xi)$$

where $$\xi = a + \theta(b-a).$$

These six definitions say in different words that the function given is continuous in a closed region, and that its derivative exists (finite or if infinite of fixed sign) in the open interval. The form of equation given by Hobson is sometimes more convenient than the other forms of equation, since then it can be seen at once whether the point $x+\theta h$ lies between a and b or not, since in that case θ must be a proper fraction between 0 and 1.

3. PROOF OF THE LAW OF THE MEAN.

For our purpose, we shall state the Law of the Mean in the following form:

THEOREM I. *If $f(x)$ is a function continuous in the interval (a,b), end points included, and has a derivative (finite or infinite) at every point within the interval, end points not included, then there is at least one point c lying between a and b such that*

$$f(b) - f(a) = (b-a)\cdot f'(c).$$

The proof of this theorem is based on Rolle's Theorem, which may be stated as follows:

THEOREM II. *Let $f(x)$ be continuous in the interval $a \leqq x \leqq b$, and let*

$$f(a) = f(b).$$

Then there exists a point c within the interval for which

$$f'(c) = 0. \qquad a < c < b.$$

Since the function $f(x)$ is continuous in the closed interval, it must be finite in the interval and its upper and lower bounds must therefore be finite and different from the end points except in the special case of a constant value of the function. Then the function must take on its upper and lower bounds at some point c within the interval. (1) Then in the neighborhood of the point c,

---

(1) Pierpont. *Theory of Functions.* pp. 216ff.

we have

$$f(c+h) - f(c) \leqq 0, \quad f(c-h) - f(c) \leqq 0.$$

Hence,

$$\frac{f(c+h) - f(c)}{h} \leqq 0, \tag{1}$$

and

$$\frac{f(c-h) - f(c)}{-h} \geqq 0. \tag{2}$$

The inequality (1) gives in the limit as h becomes indefinitely small

$$f'(c) \leqq 0; \tag{3}$$

and the inequality (2) gives in the limit as h becomes indefinitely small

$$f'(c) \geqq 0. \tag{4}$$

The two inequalities (3) and (4) express the condition, since the quantity $f'(c)$ cannot have two values at the point c and hence cannot be both greater than and less than zero, that

$$f'(c) = 0.$$

This proves Rolle's Theorem.

To prove the Law of the Mean, we set up arbitrarily the function

$$\varphi(x) = f(b) - f(x) - \frac{f(b) - f(a)}{b - a} \cdot (b-x).$$

In this function $\varphi(a) = \varphi(b) = 0$. At the points where $f'(x)$ is finite

$$\varphi'(x) = -f'(x) + \frac{f(b) - f(a)}{b - a}.$$

At the other points of the interval, $\varphi'(x)$ is infinite. Since $f(x)$ is continuous within the interval, $\varphi(x)$ is continuous within the interval and $\varphi'(x)$ is finite or infinite within the interval; thus $\varphi(x)$ satisfies the conditions of Rolle's Theorem. Whence, for some point c within the interval

$$\varphi'(c) = 0.$$

Therefore substituting c for x in $\varphi'(x)$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which was to be proved.

This result may be written in another form by substituting the values $b = x+h$, $a = x$, and $c = x+\xi$ in the equation.

We have then

$$f(x+h) - f(x) = h \cdot f'(x+\xi).$$

It is evident that $x+\xi$ lies between the points x and $x+h$. That is, ξ is less in absolute value than the absolute value of h, and it is a function of x and h.

4. EXAMPLES SHOWING THE CONDITIONS OF THE LAW OF THE MEAN NOT NECESSARY.

It may be shown that the conditions given in the Theorem of the Mean are sufficient but not necessary; that is, the Law of the Mean holds for functions when these conditions are not satisfied. Consider, for example, the function

$$f(x) = \sin(1/x).$$

This function is discontinuous at the origin and yet the Law of the Mean holds as may be seen from a geometrical consideration. For, as is shown in Figure 1, the above function oscillates between the lines $x=+1$ and $x=-1$ becoming dense at the origin. And in each loop, there is a point at which the tangent to the curve is parallel to a secant line, even though the secant line cuts the curve in points lying on opposite sides of the origin.

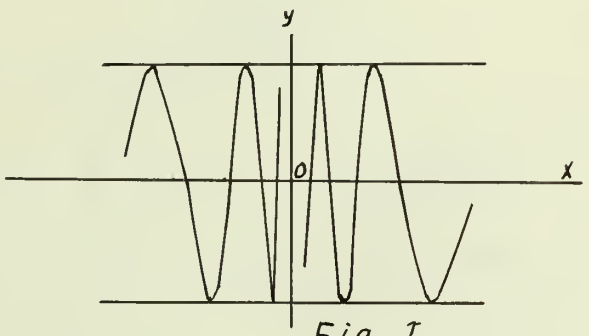

Fig. 1

The analytic proof of the preceding paragraph is as follows: From the Law of the Mean, we have

$$\sin\frac{1}{x+h} - \sin\frac{1}{x} = -h \cdot \frac{1}{(\xi+x)^2}\cos\frac{1}{x+\xi}.$$

We select arbitrarily the values $x=-1/\pi$, $h=2/\pi$ in order to have an interval including the origin. From the above equation, we have

$$\sin(\pi) - \sin(-\pi) = \frac{2/\pi}{(\xi-1/\pi)^2} \cdot \cos\frac{\pi}{\pi\xi-1}.$$

But since $\sin(\pi) - \sin(-\pi) = 0$, we obtain

$$\cos\frac{\pi}{\pi\xi-1} = 0. \qquad (1)$$

Solving this equation for ξ, we have

$$\frac{\pi}{\pi\xi-1} = n\pi + \frac{\pi}{2} = \frac{2n\pi+\pi}{2},$$

whence,

$$\pi\xi-1 = \frac{2}{2n+1}.$$

This relation gives

$$\xi = \frac{2n+3}{(2n+1)\pi}.$$

Consequently, we obtain for

$$n = 1, 2, 3, \cdots, n, \cdots$$
$$\xi = \frac{5}{3\pi}, \frac{7}{5\pi}, \frac{9}{7\pi}, \cdots, \frac{2n+3}{(2n+1)\pi}, \cdots, \frac{1}{\pi}.$$

For this interval, n corresponds to the number of oscillations of the curve counting from the upper end point of the interval.

This solution of equation (1) gives only the points lying in the positive portion of the interval. For the negative portion of the interval, we obtain similar results. In solving equation (1) we let

$$\frac{\pi}{\pi\xi-1} = -(n\pi+\frac{\pi}{2}) = -(\frac{2n\pi+\pi}{2}),$$

whence

$$\xi = \frac{2n-1}{(2n+1)\pi}.$$

From this relation, we obtain for

$$n = 1, 2, 3, \cdots, n, \cdots$$
$$\xi = \frac{1}{3\pi}, \frac{3}{5\pi}, \frac{5}{7\pi}, \cdots, \frac{2n-1}{(2n+1)\pi}, \cdots, \frac{1}{\pi}.$$

For this interval, n corresponds to the number of oscillations of the curve counting from the lower end point of the interval.

This example shows that although a function is discontinuous at a point of an interval, still the Law of the Mean holds. In this case, the point of discontinuity is the origin and a tangent line can be drawn parallel to a secant line at an infinite number of points of the interval, dense at the origin.

As another example of a function showing the Law of the Mean to hold even though the conditions of the theorem are not completely satisfies, we give the following function which is continuous but whose derivative does not exist at the origin, and yet the Law of the Mean holds for an interval including that point.

$$f(x) = x\cdot\sin(1/x)\ , \qquad x \neq 0,$$
$$= 0\ , \qquad x = 0.$$
$$f'(x) = \sin(1/x) - (1/x)\cdot\cos(1/x), \qquad x \neq 0,$$
$$= \lim_{h=0} \sin(1/h), \qquad x = 0.$$

But this limit does not exist and therefore the derivative of the function does not exist at the origin.

We apply the equation of the Law of the Mean to the given function, and have

$$(x+h)\cdot\sin\frac{1}{x+h} - x\cdot\sin(\frac{1}{x}) = h[\sin\frac{1}{x+\xi} - \frac{1}{x+\xi}\cdot\cos\frac{1}{x+\xi}]. \qquad (1)$$

Consider an interval which includes the origin, namely:

$$(x = -\frac{1}{\pi},\ h = \frac{2}{\pi}).$$

Substituting these values of x and h in equation (1), we have

$$\frac{1}{\pi}\cdot\sin(\pi) + \frac{1}{\pi}\cdot\sin(-\pi) = \frac{2}{\pi}[\sin\frac{1}{\xi-\frac{1}{\pi}} - \frac{1}{\xi-\frac{1}{\pi}}\cos\frac{1}{\xi-\frac{1}{\pi}}]. \qquad (2)$$

The left hand member of (2) is zero, giving the equation

$$\sin\frac{\pi}{\pi\xi-1} - \frac{\pi}{\pi\xi-1}\cos\frac{\pi}{\pi\xi-1} = 0. \qquad (3)$$

The equation (3) reduces to the form

$$\tan\frac{\pi}{\pi\xi-1} = \frac{\pi}{\pi\xi-1}. \qquad (4)$$

Equation (4) is an equation which is satisfied for values of $\frac{\pi}{\pi\xi-1}$ lying at the intersection of the line $x=y$ and the curve of the trigonometric tangent. From these values of $\frac{\pi}{\pi\xi-1}$ the values of ξ may be found which satisfy the restrictions imposed upon ξ by the Law of the Mean. We find for ξ a set of values lying within the interval $(-\frac{1}{\pi},\ \frac{2}{\pi})$ and dense at the origin. This result agrees with the result of the geometrical interpretation of the Law of the Mean applied to the function in question.

In figure 2 the curve is shown. It oscillates between the lines bisecting the first and third quadrants and the second and fourth quadrants. The curve is continuous at the origin but its derivative does not exist at that point. It is evident from the figure that tangent lines may be drawn at the maximum points of each loop of the curve parallel to the x-axis. Also these points

become dense at the origin.

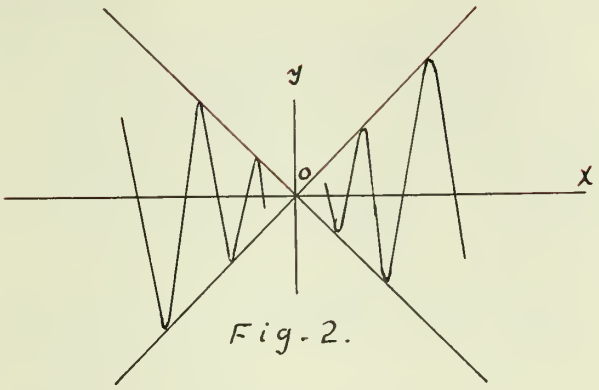

Fig. 2.

5. EXTENSIONS OF THE LAW OF THE MEAN.

THEOREM III.[1] *Suppose a function $f(x)$ is finite and continuous in a given interval $(a \leqq x \leqq b)$ and has a derivative in all points except at most in a finite number of points $a_1, a_2, a_3, \ldots$ in which it is undefined, but in all other points remains always less than a chosen finite number. Suppose also that for the points $a_1, a_2, a_3, \ldots$ the ratios*

$$\frac{f(a_1+\delta)-f(a_1)}{\delta}, \quad \frac{f(a_2+\delta)-f(a_2)}{\delta}, \quad \frac{f(a_3+\delta)-f(a_3)}{\delta}, \quad \ldots$$

approach limits not finite and defined but oscillate between two finite numbers. We have then

$$f(x+h) - f(x) = hA,$$

for x and $x+h$ lying in the given interval and where A is a number independent of x and h and always in absolute value less than a given finite positive number.

If $f(x)$ has everywhere between x and $x+h$ a definite derivative, this theorem follows at once from the Law of the Mean, since no point $a_1, a_2, a_3, \ldots$ would lie between x and $x+h$.

If one or more of the points $a_1, a_2, a_3, \ldots$ lie between x and $x+h$, we have by hypothesis

$$\left|\frac{f(a_1+\delta)-f(a_1)}{\delta}\right| < g, \left|\frac{f(a_2+\delta)-f(a_2)}{\delta}\right| < g, \ldots \quad |\delta| \leqq |\delta_1|$$

g finite.

Also if g' is the maximum value of $f(x)$ in the given interval

$$\left|\frac{f(a_1+\delta)-f(a_1)}{\delta}\right| < \frac{2g'}{\delta_1}, \left|\frac{f(a_2+\delta)-f(a_2)}{\delta}\right| < \frac{2g'}{\delta_1}, \ldots |\delta| \leqq |\delta_1|$$

---

(1) Dini. *Theorie der Functionen einer veränderlichen reelen Grösse.*

Let A' be the greater of the two numbers g and $\frac{2g'}{\delta_1}$. We assume that at least one of the points $a_1, a_2, a_3, \ldots$, say a_1 lies in the interval from x to $x+h$. That is, we assume that

$$x = a_1 \mp \delta \ , \quad x+h = a_1 \pm \delta' \tag{1}$$

where δ and δ' are positive and $\delta+\delta' = h$. Collecting the foregoing results, we obtain

$$|f(a_1 \pm \delta') - f(a_1)| < \delta' A', \ |f(a_1 \mp \delta) - f(a_1)| < \delta A',$$

whence

$$|f(a_1 \pm \delta') - f(a_1 \mp \delta)| < (\delta+\delta')A'. \tag{2}$$

Combining (1) with (2), we have

$$f(x+h) - f(x) = hh_1A' = hA. \qquad |h_1| < 1.$$

THEOREM IV. CAUCHY'S THEOREM. *Let $f(x)$ and $F(x)$ be continuous in the closed interval $(a \leqq x \leqq b)$ and let $f'(x)$ exist and be finite or infinite in the open interval $(a<x<b)$ and let $F'(x)$ be finite and different from zero in $(a<x<b)$. Then*

$$\frac{f(x+h) - f(x)}{F(x+h) - F(x)} = \frac{f'(x+\xi)}{F'(x+\xi)} \ . \tag{3}$$

It is evident that

$$F(x+h) \neq F(x).$$

For if $F(x+h) = F(x)$, we can apply Rolle's Theorem and obtain $F'(x+\xi_1) = 0$ for some point within the interval which is contrary to hypothesis.

To prove (3), we introduce the auxiliary function [1]

$$\varphi(x) = f(x) - f(a) = \frac{f(b) - f(a)}{F(b) - F(a)} \cdot [F(x) - F(a)],$$

$\varphi(x)$ is obviously finite and continuous in (a,b) since it is the sum of a finite number of continuous functions. And for $f'(x)$ finite

$$\varphi'(x) = f'(x) = \frac{f(b) - f(a)}{F(b) - F(a)} \cdot F'(x),$$

while for the other points within (a,b), $\varphi'(x)$ is infinite and of definite sign. We observe also that

$$\varphi(a) = \varphi(b) = 0.$$

Thus the conditions of Rolle's theorem are completely filled and by applying it we obtain

$$\varphi'(x+\xi) = f'(x+\xi) - \frac{f(b) - f(a)}{F(b) - F(a)} \cdot F'(x+\xi) = 0.$$

---

(1) Pierpont. *Theory of Functions of Real Variables.* p. 300.

THEOREM V. [1] *If $f(x)$ is continuous in the closed interval $(a \leqq x \leqq b)$ and if $f'(x)$ exists (finite or infinite) in the open interval $(a<x<b)$, then*

1) The derivative $f'(x)$ assumes every value between its upper and lower bounds at points lying within the interval.

2) The upper bounds of the derivative $f'(x)$ are the same as those of the incrementary ratio within the interval.

We define the incrementary ratio as

$$Q(x,h) = \frac{f(x+h) - f(x)}{h} ;$$

and the derivative as

$$f'(x) = \lim_{h \doteq 0} Q(x,h).$$

Let U and L denote the upper and lower bounds of $f'(x)$. We must show that for any number μ lying between U and L there exists a point x_0 $(a<x_0<b)$ such that

$$f'(x_0) = \mu.$$

To prove 1), we consider the auxiliary function [2]

$$\varphi(x) = f(x) - \mu x - \nu,$$

where μ and ν are finite constants and

$$L < \mu < U.$$

Suppose first that U and L are finite and that at the points $x = \alpha$, and $x = \beta$, situated within the given interval (a,b), we have, supposing that $\alpha < \beta$,

$$f'(\alpha) = L \; ; \; f'(\beta) = U.$$

Then for a sufficiently small positive number h, we have

$$\lim_{h \doteq 0} \frac{\varphi(\alpha+h)-\varphi(\alpha)}{h} = \lim_{h \doteq 0} \frac{f(\alpha+h)-\mu\cdot(\alpha+h)-\nu-f(\alpha)+\mu\cdot\alpha+\nu}{h}$$

$$= \lim_{h \doteq 0} \frac{f(\alpha+h)-f(\alpha)-\mu\cdot h}{h} = f'(\alpha)-\mu = L-\mu < 0.$$

Similarly,

$$\lim_{h \doteq 0} \frac{\varphi(\beta-h)-\varphi(\beta)}{-h} = \lim_{h \doteq 0} \frac{f(\beta-h)-f(\beta)+\mu\cdot h}{-h} = U+\mu > 0.$$

These relations show that $\varphi(x)$ is decreasing as $x \doteq \alpha$ and decreasing as $x \doteq \beta$, and that it is not a minimum at either α or β, since in that case the limits would be zero.

---

(1). Young. *Quarterly Journal of Pure and Applied Math. 1908-9.* p.11
(2) Dini. *Lezioni di Analisi Infinitesimale.* p.55.

Since $\varphi(x)$ is a continuous function, there must be a point, say $x = x_0$, lying between α and β at which φ is a minimum. Therefore, we have

$$\varphi(x_0+h) - \varphi(x_0) \geq 0; \quad (x_0-h) - \varphi(x_0) \geq 0,$$

whence,

$$\frac{\varphi(x_0+h) - \varphi(x_0)}{h} \geq 0; \quad \frac{(x_0-h) - \varphi(x_0)}{h} \leq 0.$$

From these two inequalities, we have in passing to the limit as h approaches 0, since $\varphi'(x_0)$ exists and cannot be positive and negative at the same time,

$$\varphi'(x_0) = 0,$$

whence from (1), we have

$$f'(x_0) = \mu.$$

But since μ is any number between L and U, $f'(x)$ takes under the assumed conditions any value between L and U for points lying within the given interval.

In the case where α is greater than β, the same reasoning would apply and we should find that $f'(x)$ takes on the value μ for some point lying within the interval.

In case $f'(x)$ does not take on its upper and lower bounds, it will take on values in the neighborhood of U and L if U and L are finite and hence will assume any value μ lying between U and L. If either or both U and L are infinite (that is, $U=+\infty$ and $L=\mp\infty$), $f'(x)$ will take on values as large as we wish and the theorem 1) is completely proved.

To prove 2), let U' and L' be the upper and lower bounds of the incrementary ratio $Q(x,h)$. Therefore, we may write

$$L' \leq Q(x,h) \leq U'.$$

Then since

$$f'(x) = \lim_{h=0} Q(x,h),$$

we have

$$L' \leq f'(x) \leq U'.$$

This shows L' and U' to be the upper and lower bounds of the derived function $f'(x)$.

6. OUTLINE OF PAPER BY E.R. HEDRICK, ANNALS OF MATHEMATICS, JULY, 1906.

In the *Annals of Mathematics* for July, 1906, Professor E.R. Hedrick published an article under the title *"On a Function which Occurs in the Law of the Mean"*. In this article, the author discussed limits of a function over an assemblage of points and then made a careful study of the Law of the Mean and in particular of the function $\xi(x,h)$ which occurs in the Law of the Mean and its relation to the derivative $f'(x)$. The Law of the Mean was stated in the form:

$$f(x+h) - f(x) = h \cdot f'(x+\theta h) = h \cdot f'(x+\xi), \qquad 0<\theta<1$$

Among the theorems proved in Professor Hedrick's article, are the following:

THEOREM VI. *The sufficient condition that $f'(x)$ be continuous is that $\xi(x,h)$ be continuous when x is constant.*

For, suppose ξ is continuous at $x=k$. We have for an arbitrarily chosen small number ε,

$$|\xi(k,h) - \xi(k,h_0)| < \varepsilon, \qquad h < \rho.$$

From the Law of the Mean, we have

$$\frac{f(x+h) - f(x)}{h} = Q(x,h) = f'(x+\xi).$$

Also from the definition of $f'(x)$, we have

$$|Q(k,h) - f'(k)| < \varepsilon_i, \qquad |h| < \rho_i. \qquad (1)$$

But from the Law of the Mean and the continuity of ξ, we must have $|\xi| < |h|$. Now let ρ decrease through a set of values

$$\rho_1, \rho_2, \cdots, \rho_n, \cdots \qquad (\lim_{n=\infty} \rho_n = 0)$$

and while ρ decreases let ε decrease through a set of values

$$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots \qquad (\lim_{n=\infty} \varepsilon_n = 0)$$

At the same time, as ρ decreases ξ must decrease remaining always less than h and hence becoming zero. We now replace $Q(k,h)$ in (1) by its value $f'(k+\xi)$ and rewrite (1) as

$$|Q(k,h) - f'(k)| \equiv |f'(k+\xi) - f'(k)| < \varepsilon_i,$$

which expresses the condition that $f'(x)$ is continuous at $x=k$.

The function $Q(x,h)$ approaches $f'(x)$, if $f'(x)$ exists, over any sequence of h's whose limit is zero. Let such a sequence be

$$h_1, h_2, h_3, \ldots, h_n, \ldots$$

h_n being independent of x. $f'(x)$ can evidently be represented as the

limit of the convergent series

$$Q(x,h_1) + [Q(x,h_2)-Q(x,h_1)] + [Q(x,h_3)-Q(x,h_2)] + \ldots + [Q(x,h_n)-Q(x,h_{n-1})] + \ldots \qquad (2)$$

Since $Q(x,h)$ is continuous for h different from zero, each term of the series (2) is a continuous function of x. This leads to the theorem:

THEOREM VII. *The sum $f'(x)$ of the series (2) of continuous functions is continuous in an interval when and only when the series converges uniformly in that interval.*

$$Q(x,h) = \frac{f(x+h) - f(x)}{h}$$

is continuous in x for $h \neq 0$, since $f(x)$ is continuous in the closed interval (a,b). We have the series (2)

$$f'(x) = Q(x,h_1) + [Q(x,h_2)-Q(x,h_1)] + [Q(x,h_3)-Q(x,h_2)] + \ldots + [Q(x,h_n)-Q(x,h_{n-1})] + \ldots \qquad \lim_{n=\infty} h_n = 0, \quad (2)$$

which is a series whose terms are continuous functions of x. If the series (2) converges uniformly in the given interval, the function $f'(x)$ defined by the series is also a continuous function of x. For, let S_n be the sum of the first n terms. Since the series is uniformly convergent, we have

$$|f'(x) - S_n(x)| < \varepsilon, \qquad n \geq m: \ a \leq x \leq b. \quad (3)$$

At any point x_0 then, we have

$$|f'(x_0) - S_n(x_0)| < \varepsilon, \qquad a \leq x_0 \leq b. \quad (4)$$

Since each term of (2) is continuous and we have only a finite number of terms in S_n, we have

$$|S_n(x) - S_n(x_0)| < \varepsilon, \qquad x - x_0 < \delta. \quad (5)$$

Adding (3), (4), and (5) we obtain the condition for the continuity of $f'(x)$, namely,

$$|f'(x) - f'(x_0)| < 3\varepsilon.$$

This proves the first part of the theorem; namely, that the function $f'(x)$ is continuous when the series (2) converges uniformly in the given interval.

To prove the remainder of the theorem, Professor Hedrick assumes $f'(x)$ continuous and proves that the series converges uniformly as a result of the continuity of $f'(x)$. For since any continuous function is uniformly continuous, $f'(x+t)$ approaches $f'(x)$ uniformly. Also

from the Law of the Mean

$$Q(x, h_i) = f'(x+\xi_i)$$

and $f'(x+\xi_i)$ must approach $f'(x)$ uniformly, since

$$|f'(x) - f'(x+\xi_i)| < \varepsilon, \quad i \geq N, \text{independent of } x$$

This gives then upon substituting for $f'(x+\xi_i)$,

$$|f'(x) - Q(x, h_i)| < \varepsilon, \quad i \geq N, \text{independent of } x.$$

This last inequality expresses the condition that the series (2) converges uniformly in the given interval.

Uniform convergence by segments also gives a necessary and sufficient condition for the continuity of $f'(x)$. This condition can be stated in the form of a theorem, as follows:(1)

THEOREM VIII. *If the derivative $f'(x)$ exists for every value of x in the closed interval (a,b), a necessary and sufficient condition that $f'(x)$ be continuous is that to an arbitrarily small positive number σ and to an arbitrary $h_i = h_0$, there shall correspond a positive number l and a finite set of values*

$$h = h_1, h_2, \ldots, h_n,$$

lying between h_i and h_0, for which the following condition holds: On the lines $h = h_n$ $(n = 1, 2, \ldots, n)$, it shall be possible to choose a series of segments, each of length at least equal to l, whose projections on the line $h = h_0$, when taken together, completely cover the interval $a \leq x \leq b$, and for which, furthermore, the relation

$$|Q(x, h_0) - Q(x, h_n)| < \sigma$$

holds, (x, h_n) being an arbitrary point on any one of these segments.

This condition is necessary. For, assume $f'(x)$ continuous in $a \leq x \leq b$. Throughout the discussion we shall use h_0 as the limiting point instead of the value $h = 0$. For any x' in the given interval and for $h_n - h_0 < \delta$ there exists a neighborhood $(x' - \delta_{h_n}, x' + \delta_{h_n})$, which may vary with h_n but always in such a manner that for each x within it, the inequality

$$|Q(x, h_0) - Q(x, h_n)| < \sigma \qquad (1)$$

holds for any arbitrarily chosen σ. Call the part of the interval on the line $h = 0$ on the right of x', $\Delta(x', h_n)$ and that on the left

---

(1) E. J. Townsend. *Bulletin of American Math, Society, 1905.*

$\Delta'(x, h_n)$. For some of the values of h_n, say h_m, in the set

$$h = h_1, h_2, \ldots, h_n,$$

$\Delta(x', h_n)$ may be zero and consequently for $h = h_n$, no neighborhood to the right x' exists such that (1) holds. But $\Delta(x', h_n)$ cannot be zero for all values of h_n, in the set, unless x' is the extreme point b of the interval; for we know that when h_n is taken sufficiently near h_0, the interval $\Delta(x', h_n)$ is always greater than zero. Let $\Delta(x')$ be the upper limit of the values of $\Delta(x', h_n)$. Similarly, there exists an upper limit $\Delta'(x')$ for $\Delta'(x', h_n)$ lying on the left of x'. It can be shown that the lower limit l of

$$\Delta'(x) + \Delta(x)$$

is always greater than zero. This sum is uniquely determined for each value x' of x, where $a \leq x \leq b$. Let x_1 be a point such that

$$\Delta(x_1) + \Delta'(x_1) = l$$

in every point in the neighborhood of x_1. At the point x_1, we have

$$\Delta(x_1) + \Delta'(x_1) > 0.$$

Between h_n and h_0, there is at least one value of h, say h_{m_1}, for which $\Delta(x_1, h_{m_1})$, within which for each value of x we have

$$\left| f(x, h_0) - f(x, h_{m_1}) \right| < \sigma,$$

is as near $\Delta'(x_1)$ as we choose.

On the line $h = h_0$, consider the subinterval

$$\left[x_1 - \frac{\Delta'(x_1, h_{m_2})}{2}, \; x_1 + \frac{\Delta(x_1, h_{m_1})}{2} \right].$$

For every value x'' in this sub-interval, there exists the neighborhood

$$\left[x_1, \; x_1 + \frac{\Delta(x_1, h_{m_1})}{2} \right]$$

or the neighborhood

$$\left[x_1 - \frac{\Delta'(x_1, h_{m_2})}{2}, \; x_1 \right),$$

according as x'' is on the right or left of x_1, such that for any value of x in these neighborhoods, we have either

$$\left| f(x, h_0) - f(x, h_{m_1}) \right| < \sigma$$

or

$$\left| f(x, h_0) - f(x, h_{m_2}) \right| < \sigma.$$

Consequently, for every value of x'', the corresponding value of $\Delta'(x) + \Delta(x)$ is greater than the smaller of the two numbers

$$\frac{\Delta'(x, h_{m_2})}{2}, \quad \frac{\Delta(x, h_{m_1})}{2}.$$

Therefore the lower limit of $\Delta'(x) + \Delta(x) = l$ must be greater than or at most equal to the smaller of these same two numbers. But these numbers may be made to differ as little as we please from

$$\frac{\Delta'(x_1)}{2}, \quad \frac{\Delta(x_1)}{2}$$

respectively, and since these are greater than zero, l must be greater than zero.

Since the lower limit of $\Delta'(x) + \Delta(x)$ is greater than zero, a finite number of segments lying between $h = h_n$ and $h = h_0$ can be found fulfilling the requirements of the theorem. The projections of these segments on $h = h_0$ will in general overlap. A finite number of segments is sufficient. For, if there are an infinite number, there must be an infinite number of end points projected on $h = h_0$ and hence at least one limiting point, say $x = x_0$. But $\Delta'(x) + \Delta(x) = l$ for $x = x_0$ and hence by a proper selection of a segment, we would have a neighborhood on $h = h_0$ equal to or greater than l and free from end points. But this is contrary to the supposition that the end points are dense at $x = x_0$ and hence only a finite number of segments satisfy the conditions of the theorem.

Hence, having selected σ and h_n, we may choose $l > 0$ and lay off on the lines $h = h_1, h_2, \ldots, h_n$ between h_n and h_0, a finite number of segments filling the conditions of the theorem.

This condition is also sufficient. For, if we assume that the condition holds, we can show that it follows that $Q(x, h_0)$ is continuous and hence that $f'(x)$ is continuous in the interval $(a \leq x \leq b)$. Let x_0 be any value of x in this interval and let h_i be any value chosen from the set $[h]$ which is dense at $h = h_0 = 0$. Then by hypothesis, there exists a finite number of segments lying between h_0 and h_i filling the conditions of the theorem. Upon some one of these lines, say h_κ $(\kappa < i)$, there exists for x_0 a neighborhood for which

$$\left| Q(x, h_0) - Q(x, h_\kappa) \right| < \sigma.$$

Again, if we put $h_i = h_{i_2}$ where h_{i_2} lies between h_0 and h_κ, there must also exist a finite number of segments lying between $h = h_{i_2}$ and $h = h_0$ satisfying the conditions of the theorem. Upon some one of

these lines, say $h = h_{K_2}$, there esists for x_0 a neighborhood such that we have

$$|Q(x, h_0) - Q(x, h_{K_2})| < \sigma.$$

Continuing indefinitely in this manner, we obtain for x_0 an indefinite succession of neighborhoods, each point of which satisfies the inequality

$$|Q(x, h_0) - Q(x, h_{K_n})| < \sigma$$

where $\lim_{n=\infty} h_{K_n} = h_0$. This succession of neighborhoods satisfying the above inequality, is the condition that $Q(x, h_0)$ is continuous at the point $x = x_0$. But since x_0 is any point in the given interval, it follows that $Q(x, h_0)$ is also continuous in the same interval, and hence $f'(x)$ is also continuous in that interval.

Hence, uniform convergence by segments is also a necessary and sufficient condition for the continuity of $f'(x)$.

Baire has shown in his study of discontinuous functions [1] that the limit of any sequence of continuous functions must be continuous at least once in every interval in which the limit exists. Hence $f'(x)$ must be continuous at least once in every subinterval of (a, b). Also since $f'(x)$ assumes every value between its upper and lower bounds in an interval, if it is discontinuous at any point $x = x_0$, it takes on every value between $f'(x_0)$ and $f'(x_0) + \varepsilon$, where ε is a posiitive number such that for some x in the neighborhood of $x = x_0$, we have

$$|f'(x) - f'(x_0)| < \varepsilon.$$

The following theorem follows immediately from the preceding discussion:

THEOREM IX. *The range of values all of which a derivative $f'(x)$ actually takes on in any neighborhood of any point $x = x_0$ is zero only when the derivative is continuous at $x = x_0$.*

We have proved that a sufficient condition that $f'(x)$ be continuous is that $E(x, h)$ be a continuous function of h when x is constant. We wish to show now that the converse of this theorem is not true. We may state the converse as a theorem thus:

---

(1) Baire. Thesis: *Sur les fonctions discontinues.* p. 30 and p. 108.

THEOREM X. *It is entirely possible that $f'(x)$ exists and is continuous throughout an interval about a point $x = x_0$, and yet the assemblage of values which the function $\xi(x_0, h)$ never assumes has the cardinal number of the continuum.*

The following example is given to show that this theorem is possible:

$$y = f(x) = x^3\left(1+\sin\frac{1}{x}\right), \qquad x \neq 0,$$

$$= 0, \qquad x = 0.$$

The derivative is

$$y' = f'(x) = 3x^2\left(1+\sin\frac{1}{x}\right) - x\cos\frac{1}{x}, \qquad x \neq 0,$$

$$y' = f'(x) = \lim_{\Delta x = 0} \frac{(\Delta x)^3}{\Delta x}\left(1+\sin\frac{1}{\Delta x}\right) = 0, \qquad x = 0.$$

This derivative is continuous for all values of x including the origin. However, the function $\xi(0, h)$ does not take on all values near zero. For, it is obvious that

$$f'(x) = 0, \text{ when } x = \frac{2}{(4n-1)\pi},$$

and that

$$f(x) = 0, \text{ when } x = \frac{2}{(4n-1)\pi};$$

$$f(x) > 0, \text{ when } x \neq \frac{2}{(4n-1)\pi}.$$

Now $f'(x)$ is not identically constant, and it must be less than zero for some value of x, say x_n, which lies between two minimum values $\frac{2}{(4n-1)\pi}$ and $\frac{2}{(4n-1)\pi}$. As Fig.3 shows $f(x)$ is a continuous function oscillating between the curve $y = x^3$ and the x-axis. $\xi(0, h)$ cannot take on the value x_n. For in the equation of the Law of the Mean, we have

$$f(h) - f(0) = f'(\xi), \qquad 0 < \xi < h.$$

In this equation, the left-hand member can never be negative from the construction of the function. Hence ξ cannot take on the value x_n which value would make the right hand member negative. But x_n is any point in an interval and hence has the cardinal number of the continuum. Therefore the set of values which ξ does not assume has the cardinal number of the continuum even though $f'(x)$ is continuous. This proves the theorem.

7. ADDED PROBLEMS AND INVESTIGATIONS.

We wish now to state and prove another theorem giving the necessary and sufficient condition for the continuity of $f'(x)$. We define $Q(x,h)$ and $f'(x)$ as before.

$$Q(x,h) = \frac{f(x+h) - f(x)}{h}.$$

$$f'(x) = \lim_{h \doteq 0} Q(x,h).$$

The theorem is as follows:

THEOREM XI. *The necessary and sufficient condition that $f'(x)$ be continuous in the interval $a \leq x \leq b$, is that $Q(x,h)$ converge uniformly to $f'(x)$ in the interval (a,b).*

To prove the condition sufficient, we proceed as follows: We assume that $Q(x,h)$ converges uniformly to $f'(x)$ in the interval (a,b). That is,

$$\lim_{\substack{h \doteq 0 \\ x \doteq x_0}} Q(x,h) = f'(x_0), \quad h < \delta_0 \neq 0, \ h \text{ independent of } x, \quad a \leq x_0 \leq b.$$

$Q(x,h)$ is continuous in x for $h \neq 0$, since $f(x)$ is a continuous function. Therefore,

$$\lim_{x \doteq x_0} Q(x,h) = Q(x_0,h), \qquad h \neq 0.$$

Since $f'(x)$ exists for all values of x in the given interval, $\lim_{h \doteq 0} Q(x,h)$ must exist for all values of x in the neighborhood of x_0.

These two conditions, together with the assumption that

$$\lim_{\substack{h \doteq 0 \\ x \doteq x_0}} Q(x,h) = f'(x_0),$$

give a sufficient condition that

$$\lim_{\substack{h \doteq 0 \\ x = x_0}} Q(x,h) = \lim_{h \doteq 0, x \doteq x_0} Q(x,h) = \lim_{x \doteq x_0} \lim_{h \doteq 0} Q(x,h) = f'(x_0).$$

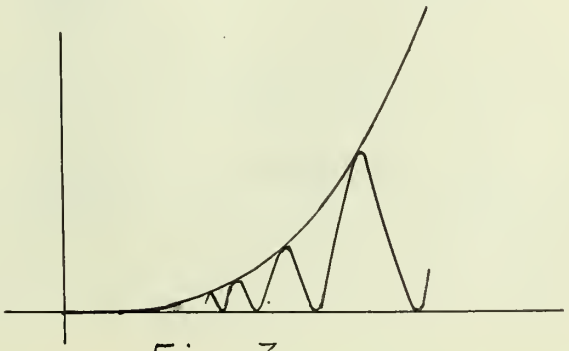

Fig. 3.

But by definition of $f'(x)$,

$$\lim_{h \doteq 0} Q(x,h) = f'(x).$$

Therefore,

$$\lim_{x \doteq x_0,\, h \doteq 0} Q(x,h) = \lim_{x \doteq x_0} f'(x) = f'(x_0).$$

This is the condition that $f'(x)$ be continnous at the point $x = x_0$. But since x_0 is any point in the interval (a,b), $f'(x)$ is continuous throughout the interval (a,b).

To prove the condition necessary, we proceed as follows: We assume $f'(x)$ continuous and prove that $Q(x,h)$ must converge uniformly to $f'(x)$ in the interval (a,b).

By assumption then, we have for an arbitrarily small positive number ε,

$$|f'(x+h) - f'(x)| < \varepsilon, \qquad |h| < \delta(x),$$

for each value of x in the interval $a \leq x \leq b$. Let $\delta_0(x)$ be the lower limit of the aggregate of δ's for all values of x in the interval (a,b). $\delta_0(x)$ cannot be zero since in that case $f'(x)$ would be discontinuous in at least ont point of (a,b) which is contrary to the hypothesis that $f'(x)$ is continuous. Therefore,

$$|f'(x+h) - f'(x)| < \varepsilon, \quad a \leq x \leq b, \quad |h| < \delta_0 \neq 0. \tag{1}$$

Also, from the definition of ξ in the Law of the Mean,

$$|f'(x+\xi) - f'(x)| < \varepsilon, \quad a \leq x \leq b, \quad |\xi| < |h| < \delta_0 \neq 0. \tag{2}$$

From the Law of the Mean,

$$Q(x,h) = f'(x+\xi), \qquad h \text{ arbitrary}, \ \xi < h. \tag{3}$$

Let $h = h_i \leq \delta_0$, $h_i \neq 0$ but $\lim_{i \doteq \infty} h_i = 0$.

Then, for any x in (a,b), we have from (2) and (3),

$$|f'(x) - Q(x,h_i)| < \varepsilon,$$

h_i independent of x and $h_i < \delta_0$. Therefore,

$$\lim_{\substack{x \doteq x_0 \\ h_i \doteq 0}} Q(x,h_i) = f'(x_0), \qquad a \leq x_0 \leq b.$$

This relation is the condition that $Q(x,h)$ approaches $f'(x)$ uniformly, and the theorem is proved.

In a preceding article we showed that the Law of the Mean may hold even though the given conditions are not satisfied at some point of the curve. We wish to show now that we may have a function which does not satisfy the conditions set forth in the theorem concerning

the Law of the Mean and indeed at every rational point, and yet the Law of the Mean holds.

We apply Hankel's method of condensation of singularities to the function

$$F(x) = x\cdot\sin\left(\frac{1}{x}\right), \qquad x \neq 0,$$
$$= 0, \qquad x = 0,$$

and obtain a function which is continucus and whose derivative does not exist at the rational points. Replacing x by $\sin(n!\pi x)$, we have for $n = 1, 2, 3, \ldots$

$$\varphi_1(x) = \sin(1!\pi x)\cdot\sin\frac{1}{\sin(1!\pi x)}, \quad (x\neq 0, 1),$$
$$= 0, \qquad (x=0, 1)$$

We define $\varphi_1(x)$ as zero for $x = 0$ and $x = 1$ since

$$\sin(\pi) = \sin(0) = 0$$

and $\sin\frac{1}{\sin(\pi)}$ and $\sin\frac{1}{\sin(0)}$ cannot be than unity. We have also,

$$\varphi_2(x) = \sin(2!\pi x)\cdot\sin\frac{1}{\sin(2!\pi x)}, \quad (x\neq 0, \tfrac{1}{2}, 1),$$
$$= 0, \qquad (x=0, \tfrac{1}{2}, 1).$$
$$\varphi_3(x) = \sin(3!\pi x)\cdot\sin\frac{1}{\sin(3!\pi x)}, \quad (x\neq 0, \tfrac{1}{3!}, \ldots, \tfrac{3!}{3!}),$$
$$= 0, \qquad (x=0, \tfrac{1}{3!}, \ldots, \tfrac{3!}{3!}).$$
$$\cdots\cdots\cdots\cdots$$
$$\varphi_n(x) = \sin(n!\pi x)\cdot\sin\frac{1}{\sin(n!\pi x)}, \quad (x\neq 0, \tfrac{1}{n!}, \ldots, \tfrac{n!}{n!}),$$
$$= 0, \qquad (x=0, \tfrac{1}{n!}, \ldots, \tfrac{n!}{n!}).$$

$\varphi_n(x)$ is defined as zero at the rational points since $\sin(n!\pi x)$ becomes the sine of an integral multiple of π and is therefore zero, and the second factor of $\varphi_n(x)$ cannot be greater than unity from the nature of the sine function.

We define

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{(n+2)!}\cdot\varphi_n(x) = \sum_{n=1}^{\infty}\frac{1}{(n+2)!}\cdot\sin(n!\pi x)\cdot\sin\frac{1}{\sin(n!\pi x)}.$$

The function $\varphi_n(x)$ is continuous. For,

$$\varphi_n(x+\varepsilon)-\varphi_n(x)=\sin[n!\pi(x+\varepsilon)]\cdot\sin\frac{1}{\sin[n!\pi(x+\varepsilon)]} - \sin(n!\pi x)\cdot\sin\frac{1}{\sin(n!\pi x)}.$$

Expanding $\sin[n!\pi(x+\varepsilon)]$ as the sine of the sum of two angles, we have,

$$\sin[n!\pi(x+\varepsilon)] = \sin(n!\pi x)\cdot\cos(n!\pi\varepsilon) + \cos(n!\pi x)\cdot\sin(n!\pi\varepsilon).$$

Passing to the limit as $\varepsilon \doteq 0$, we have since the sine function is continuous,

$$\lim_{\varepsilon \doteq 0} \sin[n!\pi(x+\varepsilon)] = \sin(n!\pi x),$$

since $\lim_{\varepsilon \doteq 0} \cos(n!\pi\varepsilon) = 1$ and $\lim_{\varepsilon \doteq 0} \sin(n!\pi\varepsilon) = 0$. We are now able to find the limit of $\varphi_n(x+\varepsilon) - \varphi_n(x)$. applying the laws of limits, we obtain

$$\lim_{\varepsilon \doteq 0} \varphi_n(x+\varepsilon) - \varphi_n(x) = \lim_{\varepsilon \doteq 0} \sin[n!\pi(x+\varepsilon)] \cdot \sin \frac{1}{\lim_{\varepsilon \doteq 0} \sin[n!\pi(x+\varepsilon)]}$$

$$- \sin(n!\pi x) \cdot \sin \frac{1}{\sin(n!\pi x)}$$

$$= \sin(n!\pi x) \cdot \sin \frac{1}{\sin(n!\pi x)}$$

$$- \sin(n!\pi x) \cdot \sin \frac{1}{\sin(n!\pi x)}$$

$$= 0.$$

This equation expresses the condition that $\varphi_n(x)$ is a continuous function of x. That is, every term in the series defining the function $f(x)$ is a continuous function of x. Also, since $\varphi_n(x)$ is the product of two sines, it can never be greater than unity. Hence, every term of $f(x)$ is not greater than the corresponding term of the series $\Sigma \frac{1}{(n+2)!}$ which is a convergent series. Therefore by the Weierstrass test for uniform convergence, $f(x)$ is a uniformly convergent series and the terms being all continuous, it is also a continuous function.

We wish now to consider the derivative of the function $f(x)$. We have:

$$\frac{d[\varphi_n(x)]}{dx} = n!\pi \cdot \cos(n!\pi x) \cdot \left[\sin \frac{1}{\sin(n!\pi x)} - \frac{1}{\sin(n!\pi x)} \cdot \cos \frac{1}{\sin(n!\pi x)}\right],$$

for $x = 0, \frac{1}{n!}, \frac{2}{n!}, \ldots, \frac{n!}{n!}$.

For these values of x, the cosine cannot be unity, nor the sine zero, hence,

$$\varphi_n'(x) < n!\pi(1-G)$$

where G is the largest value of the product $\frac{1}{\sin n!\pi x)} \cdot \cos \frac{1}{\sin(n!\pi x)}$, and is therefore finite. If, then, we form the series of derivatives, we have

$$f'(x) = \sum_{n=1}^{\infty} \frac{\varphi_n'(x)}{(n+2)!},$$

for this series converges uniformly since each term is less than the corresponding series

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} \cdot \pi(1-G).$$

This series is convergent. For, applying Raabe's test [1], we have

$$\lim_{n=\infty} n\left(1 - \frac{v_{n+1}}{v_n}\right) = \lim_{n=\infty} n\left(1 - \frac{n+1}{n+3}\right) = \lim_{n=\infty} \frac{2n}{n+3} > 1,$$

which is the condition for convergence. Hence the series $f(x)$ may be differentiated term for term for irrational values of x, or in other words, the derivative $f'(x)$ exists for irrational values of x.

We wish now to find the derivative of $f(x)$ for rational values of x, that is, for x of the form $x = p/q$. We have then,

$$\varphi_n(x+h) - \varphi_n(x) = \varphi_n\left(\frac{p}{q}+h\right) - \varphi_n\left(\frac{p}{q}\right)$$

$$= \sin\left[n!\pi\left(\frac{p}{q}+h\right)\right] \cdot \sin\frac{1}{\sin\left[n!\pi\left(\frac{p}{q}+h\right)\right]} - \sin\left(n!\pi\frac{p}{q}\right)\sin\frac{1}{\sin\left(n!\pi\frac{p}{q}\right)}.$$

Expanding $\sin\left[n!\pi\left(\frac{p}{q}+h\right)\right]$, we have

$$\sin\left[n!\pi\left(\frac{p}{q}+h\right)\right] = \sin\left(n!\pi\frac{p}{q}\right)\cdot\cos(n!\pi h) + \cos\left(n!\pi\frac{p}{q}\right)\cdot\sin(n!\pi h)$$

$$= n!\pi h,$$

since by taking n large enough so that $n!$ is divisible by q, then $n!\pi p/q$ is an integral multiple of π and the sine of an integral multiple of π is zero and the cosine of the same angle is unity and in the limit as h becomes small, we have

$$\lim_{h=0} \sin(n!\pi h) = n!\pi h.$$

Therefore, replacing this result in $\varphi_n(x+h) - \varphi_n(x)$, we obtain

$$\varphi_n(x+h) - \varphi_n(x) = n!\pi h \cdot \sin\frac{1}{n!\pi h}, \qquad x = \frac{p}{q},$$

and

$$\frac{\varphi_n(x+h) - \varphi_n(x)}{h} = n!\pi \cdot \sin\frac{1}{n!\pi h}, \qquad x = \frac{p}{q}.$$

This difference quotient has no limit as h approaches zero since the limit $\lim_{h=0} \sin\frac{1}{n!\pi h}$ does not exist, and hence the derivative $f'(x)$ does not exist at the rational points.

---

(1) Bromwich. *Theory of Infinite Series.* p. 38.

We have thus built up a function $f(x)$ which has the same singularities at all the rational points, as $F(x)$ had at the origin; that is, $f(x)$ is a continuous function which has a derivative at all points except at the rational points. We saw that the Law of the Mean holds for $F(x)$ over an interval including the origin. Likewise, for $f(x)$ since it is continuous and has a derivative at all irrational points, we can show that the Law of the Mean holds over any interval. For, if we select the x-axis as the secant line, we can show that the derivative $f'(x)$ vanishes at some irrational point; that is, that at some irrational point, there is a tangent to the curve $f(x)$ which is parallel to the x-axis. We see from the series defining $f'(x)$ that this must happen when

$$\sin\frac{1}{\sin(n!\pi x)} - \frac{1}{\sin(n!\pi x)}\cdot\cos\frac{1}{\sin(n!\pi x)} = 0.$$

That is, when

$$\tan\frac{1}{\sin(n!\pi x)} = \frac{1}{\sin(n!\pi x)}.$$

This equation can be solved by means of tables.

Hedrick showed in the article reviewed in Section C that we may have a function $f(x)$ which has a derivative at every point of an interval but for which the function $\xi(x,h)$ is discontinuous. We wish to show that this may be true by giving another example and by Hankel's method of condensation of singularities to extend this function to a function whose derivative exists everywhere but the function $\xi(x,h)$ is discontinuous at all the rational points. To do this we make use of the function

$$F(x) = x^2\cdot\sin(\tfrac{1}{x}), \qquad x \neq 0,$$
$$= 0, \qquad x = 0.$$

This function is continuous and its derivative is

$$F'(x) = 2x\cdot\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x}), \qquad x \neq 0,$$
$$= \lim_{h=0}\frac{h^2}{h}\cdot\sin(\tfrac{1}{x}) = 0, \qquad x = 0.$$

$F'(x)$ is discontinuous at the origin since the limit

$$\lim_{x=0}\left[2x\cdot\sin(\tfrac{1}{x}) - \cos(\tfrac{1}{x})\right]$$

does not exist. $\xi(x,h)$ is also discontinuous, for by Theorem VI, if $\xi(x,h)$ is continuous $f'(x)$ must also be continuous. This function

then has a derivative at every point and the function $\xi(x,h)$ is discontinuous at the origin.

We shall now apply Hankel's method of condensation of singularities to $F(x)$ and obtain a function which is continuous and whose derivative exists at every point and is discontinuous at the rational points. This will mean that $\xi(x,h)$ is also discontinuous at the rational points as will be seen by referring to Theorem VI.

As in the preceding example, we let

$$F(x) = x^2 \sin\left(\frac{1}{x}\right), \qquad (x \neq 0),$$
$$= 0, \qquad (x = 0),$$

and substitute $\sin(n!\pi x)$ for x. We have for $n=1,2,3,\ldots$

$$\varphi_n(x) = \sin^2(n!\pi x)\cdot \sin\frac{1}{\sin(n!\pi x)}, \quad \left(x\neq 0, \frac{1}{n!}, \frac{2}{n!}, \frac{3}{n!}, \ldots, \frac{n!}{n!}\right),$$
$$= 0, \qquad \left(x=0, \frac{1}{n!}, \frac{2}{n!}, \frac{3}{n!}, \ldots, \frac{n!}{n!}\right).$$

We define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(n+2)!}\cdot \varphi_n(x) = \sum_{n=1}^{\infty} \frac{1}{(n+2)!}\cdot \sin^2(n!\pi x)\cdot \sin\frac{1}{\sin(n!\pi x)}.$$

As before

$$\lim_{\varepsilon \doteq 0} \varphi_n(x+\varepsilon) - \varphi_n(x) = \sin^2(n!\pi x)\cdot \sin\frac{1}{\sin(n!\pi x)} - \sin^2(n!\pi x)\cdot \sin\frac{1}{\sin(n!\pi x)}.$$
$$= 0.$$

Hence $f(x)$ is defined by a series of continuous terms and also every term of the series is less than or at most equal to the corresponding term of the series $\Sigma\frac{1}{(n+2)!}$, which is a convergent series. Hence $f(x)$ is uniformly convergent, and, every term being continuous, is therefore continuous.

For irrational values of x, we have

$$\frac{d[\varphi_n(x)]}{dx} = n!\pi\cdot\cos(n!\pi x)\cdot\left[2\sin(n!\pi x)\cdot \sin\frac{1}{\sin(n!\pi x)} - \cos\frac{1}{\sin(n!\pi x)}\right].$$

We have then for $f'(x)$,

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n+2)!}\cdot \varphi_n'(x),$$

and the corresponding terms of the series defining $f'(x)$ are less than the terms of the convergent series

$$\sum_{n=1}^{\infty} \frac{n!\,\pi}{(n+2)!}.$$

Hence the derivative $f'(x)$ exists for irrational values of x.

For rational values of x, that is, for $x = p/q$, we have

$$\frac{\varphi_n(x+h) - \varphi_n(x)}{h} = \frac{1}{h}\left[\sin^2[n!\pi(\frac{p}{q}+h)]\cdot\sin\frac{1}{\sin[n!\pi(\frac{p}{q}+h)]}\right.$$

$$\left. - \sin^2(n!\pi\frac{p}{q})\cdot\sin\frac{1}{\sin(n!\pi\frac{p}{q})}\right].$$

We found before that in the limit

$$\sin[n!\pi(\frac{p}{q}+h)] = n!\pi h.$$

Substituting this value in the above differential quotient, we obtain

$$\frac{\varphi_n(x+h) - \varphi_n(x)}{h} = \frac{1}{h}\left[(n!\pi h)^2\cdot\sin\frac{1}{n!\pi h}\right.$$

Passing to the limit, we have

$$f'(x) = 0,$$

for rational values of x.

We have then, a function $f(x)$ which is continuous and which has a derivative at every point. However, we can show that this derivative is discontinuous at the rational points. For, as x approaches any rational value, the limit

$$\lim_{x=p/q} \cos\frac{1}{\sin(n!\pi x)}$$

does not exist and hence

$$\lim_{x=p/q} \frac{d[\varphi_n(x)]}{dx}$$

does not exist. In other words, $f'(x)$ is discontinuous at the rational points. We now apply Theorem VI. $\xi(x,h)$ cannot be continuous since in that case, $f'(x)$ must also be continuous. But as this is not the case, $\xi(x,h)$ must be discontinuous.

8. A CONDITION FOR THE CONTINUITY OF $\xi(h)$.

THEOREM XII. *The function $\xi(h)$ defined in the equation*

$$f(x_0+h) - f(x_0) = h\cdot f'[x_0+\xi(h)]$$

is continuous in h, if, in the interval $(0 \leq x \leq b-a)$, $\xi(h)$ is a single-valued function of h; and conversely, if $h(\xi)$ is a single-valued function of ξ.

By hypothesis, $\xi(h)$ is a single-valued function of h, and

$$\frac{f(x_0+h) - f(x_0)}{h} = f'[x_0+\xi(h)],$$

and $h(\xi)$ is a single-valued function of ξ. From the Law of the Mean, we have

$$|\xi| < |h|; \qquad (h \neq 0).$$

Also from the equation given in the theorem, and the fact that $\xi(h)$ exists in the closed interval, we have

$$\lim_{h=0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h=0} f'[x_0+\xi(h)].$$

But from the definition of a derivative, we obtain

$$\lim_{h=0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0).$$

Therefore we have,

$$\lim_{h=0} f'[x_0+\xi(h)] = f'(x_0)$$

from which, since $\xi(h)$ is single-valued, we get

$$\lim_{h=0} \xi(h) = 0.$$

This proves the continuity of $\xi(h)$ at the point $h = 0$.

To prove the continuity of $\xi(h)$ for $h \neq 0$, we assume $f(x)$ continuous for $a \leq x_0 \leq b$, and also that $f'(x_0)$ exists for $a < x_0 < b$, these conditions being the conditions given in the Law of the Mean. As h takes on all values in the interval $(0, b-a)$, ξ takes all values in some interval $(0, c)$, where $|c| < |b-a|$. ξ also satisfies the condition $|\xi| < |h|$. Corresponding to any chosen value h_0, there is one and only one value ξ_0. And conversely, to any chosen value ξ_0, there is one and only one value h_0. Also in any interval $(h_0-\delta, h_0+\delta)$, to each point in this interval, there corresponds one and only one point in an interval $(\xi_0-\delta', \xi_0+\delta')$. For, suppose that corresponding to some h' in $(h_0-\delta, h_0+\delta)$ there corresponds a point ξ' lying outside the interval $(\xi_0-\delta', \xi_0+\delta')$. Then we would have a point of discontinuity on the curve $\xi = \xi(h)$, because it is assumed to be single-valued. That is, corresponding to some ξ'' in the interval $(\xi_0-\delta', \xi_0+\delta')$, there would be no h'' in the interval $(h_0-\delta, h_0+\delta)$. But this is contrary to the hypothesis that $\xi(h)$ is single-valued. Similarly, to every ξ in the interval $(\xi_0-\delta', \xi_0+\delta')$ there corresponds

one and only one value of h lying within the interval $(h_0-\delta, h_0+\delta)$. Therefore, as h takes on all values in the interval $(h_0-\delta, h_0+\delta)$, ξ takes all values in its corresponding interval, and to each interval $(h_0-\delta, h_0+\delta)$ however small, there corresponds only one interval $(\xi_0-\delta', \xi_0+\delta')$. Now take δ' arbitrarily small. We have then

$$|\xi(h) - \xi(h_0)| < \delta'$$

for all values of h such that $|h - h_0| < \delta$. Therefore ξ must approach ξ_0 continuously, or in other words, $\xi(h)$ is a continuous function of h.

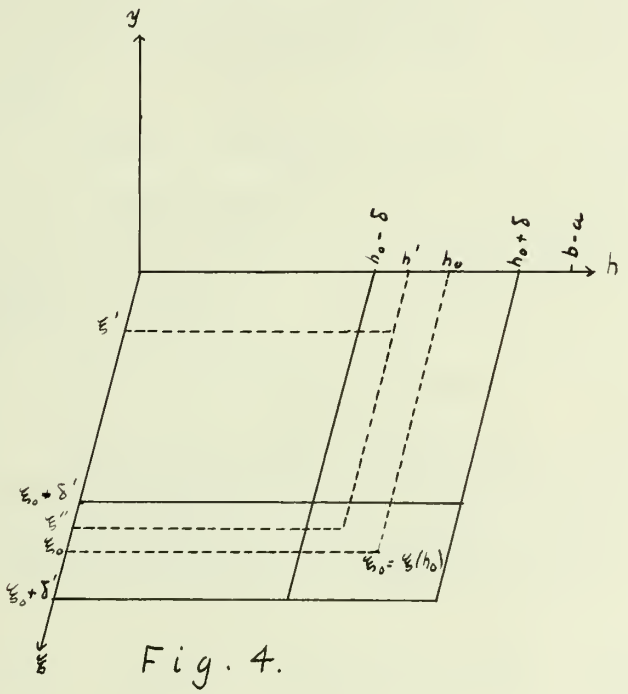

Fig. 4.

CHAPTER II.

THE LAW OF THE MEAN FOR FUNCTIONS OF TWO VARIABLES.

9. STATEMENTS OF THE FORMS OF THE THEOREM.

The Law of the Mean for functions of two variables may be written in at least three different ways, as follows:

THEOREM I. [1] *Suppose that in the closed region* $(a \leqq x \leqq a_1,\ b \leqq y \leqq b_1)$, *that* $f(x,y)$ *is a function continuous in* x *alone and in* y *alone. Suppose also that the first partial derivatives* $f'_x(x,y)$ *and* $f'_y(x,y)$ *exist in the open region* $(a<x<a_1,\ b<y<b_1)$. *Then there is at least one value for* θ_1 *and* θ_2, *lying between zero and one (zero and one excluded), such that*

$$f(x+h,y+k) - f(x,y) = h \cdot f'_x(x+\theta_1 h, y+k) + k \cdot f'_y(x, y+\theta_2 k).$$

THEOREM II. [1] *If the function* $f(x,y)$ *has first partial derivatives which are continuous in* x *alone and in* y *alone in the closed region* $(a \leqq x \leqq a_1,\ b \leqq y \leqq b_1)$, *then there must be at least one value of* θ $(0<\theta<1)$, *such that*

$$f(x+h,y+k) - f(x,y) = h \cdot f'_x(x+\theta h, y+\theta k) + k \cdot f'_y(x+\theta h, y+\theta k).$$

THEOREM III. [1] *Under the same conditions as in Theorem II, namely, that the first partial derivatives* $f'_x(x,y)$ *and* $f'_y(x,y)$ *exist and are continuous in* x *alone and in* y *alone in the closed region* $(a \leqq x \leqq a_1,\ b \leqq y \leqq b_1)$, *then there is at least one value of* θ $(0<\theta<1)$ *such that*

$$f(x+h,y+k) - f(x,y) = h \cdot f'_x(x+\theta h, y+k) + k \cdot f'_y(x, y+\theta k).$$

The advantage of the second and third statements of the Law of the Mean over the first statement, is that there is only one θ involved. However, the condition that the first partial derivatives be continuous is a little more rigid that the condition of the first statement where the first partial derivatives are assumed merely to exist. The conditions in Theorems II and III are sufficient that the function be totally differentiable in the entire region as we shall

---

(1) *Encyclopedie des Science Mathematiques.*
(2) Tannery. *Introduction, etc.*
(3) Goursat. *Cours d'Analyse.*

show in section 12. This fact will be very useful in interpreting geometrically the Law of the Mean for functions of two variables.

10. PROOFS.

1) In the first case, $f(x,y)$ is continuous in x and in y in the closed region and f'_x and f'_y exist in the open region $(a,a_1;b,b_1)$. We suppose now that $y+k$ is a constant. Then the function $f(x,y+k)$ is a function of x and its derivative exists in the region under consideration and is also a function of x. Hence we may apply the Law of the Mean for functions of one variable to the function $f(x,y+k)$ since the conditions are satisfied. Applying this theorem, we have

$$f(x+h,y+k) - f(x,y+k) = h\cdot f'_x(x+\theta_1 h, y+k). \qquad (1)$$

Similarly, if x is kept constant and y allowed to vary, $f(x,y)$ is a continuous function of y since it has a derivative which is also a function of y. Thus the conditions of the Law of the Mean for functions of one variable are satisfied. We apply the Law of the Mean to $f(x,y)$ and obtain

$$f(x,y+k) - f(x,y) = k\cdot f'_y(x,y+\theta_2 k). \qquad (2)$$

Adding equations (1) and (2), we obtain

$$f(x+h,y+k) - f(x,y) = h\cdot f'_x(x+\theta_1 h, y+k) + k\cdot f'_y(x,y+\theta_2 k).$$

Since θ_1 and θ_2 were obtained by applying the Law of the Mean for functions of one variable, they must satisfy the relations of that theorem. We must therefore have the relations

$$0 < \theta_1 < 1,\ 0 < \theta_2 < 1.$$

As an example of the application of the above theorem, we give the following:

$$f(x,y) = (x^2y^2)^{\frac{1}{4}}.$$

$$f'_x(x,y) = \frac{y}{2(x^2y^2)^{\frac{1}{4}}}, \qquad x\neq 0, y\neq 0,$$

$$= 0, \qquad x=0, y=0.$$

$$f'_x(x,y) = \frac{x}{2(x^2y^2)^{\frac{1}{4}}}, \qquad x\neq 0, y\neq 0,$$

$$= 0, \qquad x=0, y=0.$$

The partial derivatives of $f(x,y)$ exist everywhere but are discontinuous at the origin. The function itself is continuous in x alone and in y alone and hence the conditions of the first form

of the Law of the Mean are satisfied. We take a region bounded by the lines $x=0, y=0, h=1, k=1$ and apply the equation of the Law of the Mean in the first form, and obtain

$$f(1,1) - f(0,0) = f'_x(\theta_1, 1) + f'_y(0, \theta_2).$$

This reduces to the form

$$1 - 0 = \frac{1}{2(\theta_1^2)^{\frac{1}{4}}}.$$

We solve this equation for θ_1 and obtain the value

$$\theta_1 = \frac{1}{4}.$$

θ_2 may take any value between zero and one since the function is zero when $x=0$.

2) In the second case, by hypothesis $f(x,y)$ has the first partial derivatives f'_x and f'_y which exist and are continuous in x alone and in y alone in the closed region $(a, a_1; b, b_1)$.

Since f'_x and f'_y exist, $f(x,y)$ must be continuous in x alone and in y alone. Also f'_x is bounded since it exists and is continuous in x and in y in a closed region. It follows then that $f(x,y)$ converges uniformly along the line $y=y_0$. (1) Then, for any point x_0 lying on $y=y_0$ within the region, we have (1)

$$\lim_{\substack{x=x_0 \\ y=y_0}} f(x,y) = f(x_0, y_0).$$

That is, $f(x,y)$ is continuous in x and y together at any point in the given region. Let

$$x = a+ht; \quad y = b+kt, \tag{1}$$

$$\varphi(t) = f(a+ht, b+kt). \tag{2}$$

$\varphi(t)$ is therefore a function of t, where t is a function of x and y. Then since $f(x,y)$ is a continuous function of (x,y), $\varphi(t)$ must also be a continuous function of t. Also since f'_x and f'_y are continuous in x alone and in y alone, then $f(x,y)$ is totally differentiable with respect to t. For if $f(x,y)$ is a continuous function of the two variables which are htemselves functions of another variable t and if the partial derivatives f'_x and f'_y are continuous, then the derivative $df(x,y)/dt$ exists and may be expressed (2)

---

(1) E. J. Townsend. *Lectures on Theory of Functions of Real Variables.*
(2) Townsend and Goodenough. *Essentials of Calculus.* p. 237.

$$f'(x,y) = f'(a+ht, b+kt) = \varphi'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Therefore $\varphi'(t)$ must exist at every point of the region. $\varphi(t)$ thus satisfies the Law of the Mean for functions of one variable. Let $t=0$ and $\Delta t=1$ and apply the Law of the Mean to $\varphi(t)$. We obtain

$$\varphi(1) - \varphi(0) = \varphi'(\theta). \tag{3}$$

But from (2), we have

$$\varphi(1) - \varphi(0) = f(a+h, b+k) - f(a,b). \tag{4}$$

From (1), we have

$$\frac{dx}{dt} = h; \quad \frac{dy}{dt} = k,$$

whence,

$$\varphi'(t) = h\cdot f'_x(a+ht, b+kt) + k\cdot f'_y(a+ht, b+kt),$$

$$\varphi'(\theta) = h\cdot f'_x(a+\theta h, b+\theta k) + k\cdot f'_y(a+\theta h, b+\theta k). \tag{5}$$

Combining (3), (4), and (5), we obtain

$$f(a+h, b+k) - f(a,b) = h\cdot f'_x(a+\theta h, b+\theta k) + k\cdot f'_y(a+\theta h, b+\theta k).$$

If we let $a=x$, and $b=y$, we have the equation

$$f(x+h, y+k) - f(x,y) = h\cdot f'_x(x+\theta h, y+\theta k) + k\cdot f'_y(x+\theta h, y+\theta k)$$

which was to be proved.

The following example illustrates the method of applying this theorem to a function of two variables. Given the sphere

$$f(x,y) = (1-x^2-y^2)^{\frac{1}{2}}.$$

$$f'_x(x,y) = \frac{-2x}{(1-x^2-y^2)^{\frac{1}{2}}}, \qquad (x\neq 0, y\neq 0),$$

$$= 0 \qquad . \qquad (x=0, y=0).$$

$$f'_y(x,y) = \frac{-2y}{(1-x^2-y^2)^{\frac{1}{2}}}, \qquad (x\neq 0, y\neq 0),$$

$$= 0 \qquad , \qquad (x=0, y=0).$$

The first partial derivatives are continuous in x alone and in y alone and exist in the entire circular region ($|x| < 1, |y| < 1$). The conditions of the second form of the Law of the Mean are therefore filled. Applying the second form, we have for a chosen region

$$(x=0,\ y=0,\ h=\tfrac{1}{2},\ k=\tfrac{1}{2}),$$

$$\sqrt{\tfrac{1}{2}} - 1 = -\frac{\theta}{\sqrt{1-\frac{1}{2}\theta^2}}.$$

The solution of this equation gives $\theta = 1/4$ approximately which satisfies the restrictions placed on θ by the Law of the Mean.

3) The conditions for the third case being the same as for the second, we may introduce the auxiliary function

$$\varphi(t) = f(a+ht, b+k) + f(a, b+kt). \qquad (1)$$

This function is a continuous function and has a derivative at every point of the given region. Hence we may apply the Law of the Mean for functions of one variable. Letting $t=0$ and $\Delta t=1$, we obtain

$$\varphi(1) - \varphi(0) = \varphi'(\theta). \qquad 0<\theta<1. \qquad (2)$$

From (1) we may write

$$\varphi(1) - \varphi(0) = f(a+h, b+k) + f(a, b+k) - f(a, b+k) - f(a, b)$$
$$= f(a+h, b+k) - f(a, b). \qquad (3)$$

$$\varphi'(t) = f'_x(a+ht, b+k)\cdot\frac{d}{dt}(a+ht) + f'_x(a, b+kt)\cdot\frac{d}{dt}(a)$$
$$+ f'_y(a+ht, b+k)\cdot\frac{d}{dt}(b+k) + f'_y(a, b+kt)\cdot\frac{d}{dt}(b+kt)$$
$$= f'_x(a+ht, b+k)\cdot h + f'_y(a, b+kt)\cdot y. \qquad (4)$$

Replacing t by θ in (4), we have

$$\varphi'(\theta) = h\cdot f'_x(a+\theta h, b+k) + k\cdot f'_y(a, b+\theta k). \qquad (5)$$

Combining (2), (3), and (5), we have

$$f(a+h, b+k) - f(a, b) = h\cdot f'_x(a+\theta h, b+k) + k\cdot f'_y(a, b+\theta k). \qquad (6)$$

If we let $a=x$ and $b=y$, we obtain from (6) the desired form

$$f(x+h, y+k) - f(x, y) = h\cdot f'_x(x+\theta h, y+k) + k\cdot f'_y(x, y+\theta k). \qquad (7)$$

The following example illustrates the application of the third form of the Law of the Mean.

$$f(x, y) = x^3y^3\left(1+\sin\frac{1}{xy}\right). \qquad (x\neq 0, y\neq 0)$$
$$= 0 \qquad (x=0, y=0), (x=0, y\neq 0), (x\neq 0, y=0).$$

Let $x=0$, $y=0$, $h=\frac{2}{3}$, and $k=\pi$ and apply equation (7). We have

$$\left(\tfrac{2}{3}\right)^3\pi^3\left(1+\sin\frac{1}{\frac{2}{3}\pi}\right) = \tfrac{2}{3}\left[3\left(\tfrac{2}{3}\theta\right)^2\pi^3\left(1+\sin\frac{1}{\frac{2}{3}\theta\cdot\pi}\right)\right].$$

This equation reduces to the form

$$\sin\frac{3}{2\pi\theta} = -1,$$

from which

$$\frac{3}{2\pi\theta} = \frac{(4n-1)\pi}{2},$$

or

$$\theta = \frac{3}{\pi^2(4n-1)}.$$

θ thus satisfies the restrictions for an infinite number of values of n.

11. EXAMPLES SHOWING THE CONDITIONS SUFFICIENT BUT NOT NECESSARY.

$$f(x,y) = x\cdot\sin(1/y), \qquad (x\neq 0, y\neq\phi)$$
$$= 0, \qquad (x=0,y=0),(x=0,y\neq 0),(x\neq 0,y=0).$$

The function is completely defined. The derivatives are as follows:

$$f'_x(x,y) = \sin(1/y), \qquad (x\neq 0, y\neq 0).$$

$$f'_x(x,y) = \lim_{\Delta x\doteq 0}\frac{[f(\Delta x,0) - f(0,0)]}{\Delta x} = 0, \quad (x=0,y=0).$$

This derivative is discontinuous in y at the origin, since the limit

$$\lim_{y\doteq 0} f'_x(x,y) = \lim_{y\doteq 0}\sin(1/y)$$

does not exist. We have also

$$f'_y(x,y) = -\frac{x}{y^2}\cdot\cos(1/y), \qquad (x\neq 0, y\neq 0),$$

$$f'_y(x,y) = \lim_{\Delta y\doteq 0}\frac{[f(0,\Delta y) - f(0,0)]}{\Delta y} = 0, \quad (x=0,y=0)$$

This derivative is also discontinuous in y for $y=0$, for the limit

$$\lim_{y\doteq 0} f'_y(x,y) = \lim_{y\doteq 0} -\frac{x}{y^2}\cdot\cos(1/y)$$

becomes infinite.

Thus since the derivatives are discontinuous, the conditions of the second form of the Law of the Mean do not hold. However the Law of the Mean for this case does hold. For substituting in the equation of the Law of the Mean in the second form, we have

$$(a+h)\sin\frac{1}{b+k} - a\cdot\sin\frac{1}{b} = h\cdot\sin\frac{1}{b+\theta k} - k\cdot\frac{a+\theta h}{(b+\theta k)^2}\cos\frac{1}{b+\theta k}. \qquad (1)$$

Take for $a, b, h,$ and k, the values

$$a = -1, \quad b = -1/\pi, \quad h = 2, \quad k = 2/\pi.$$

These values form a region which includes the origin at which pcint the singularity occurs. We now substitute the above values in (1) and obtain,

$$\sin(\pi) - \sin(-\pi) = 2\cdot\sin\frac{\pi}{2\theta-1} - \frac{2}{\pi}\cdot\frac{(-1+2\theta)\pi}{(2\theta-1)^2}\cdot\cos\frac{\pi}{2\theta-1}. \qquad (2)$$

The left hand member of (2) being zero, the equation reduces to

$$\sin\frac{\pi}{2\theta-1} = \frac{\pi}{2\theta-1}\cdot\cos\frac{\pi}{2\theta-1}.$$

$$\tan\frac{\pi}{2\theta-1} = \frac{\pi}{2\theta-1}. \qquad (3)$$

The equation (3) is an empirical equation which can be solved by the use of tables or graphically as the intersection of the curve of the trigonometrical tangent and the line bisecting the first and third quadrants. The solution gives an infinite number of values of θ lying between zero and one. These values satisfy the restrictions placed upon θ in the Law of the Mean and show that the Law of the Mean holds for this example even though the conditions stated in the second form of the Law of the Mean are not satisfied. Hence the conditions are not necessary. They are sufficient, however, since when the conditions are satisfied the equation of the Law of the Mean in the second form must hold.

12. GEOMETRIC INTERPRETATION.

In proving the Law of the Mean for functions of two variables in the first form, we assumed that $f(x,y)$ is a continnous function in the two variables separately and that the first partial derivatives exist in the open region. From these assumptions, we can prove the following theorem: (1)

THEOREM IV. *The points where $f(x,y)$ is totally differentiable form a set of points everywhere dense, if $f(x,y)$ is defined as in the first form of the Law of the Mean.*

We define

$$F(x_0, y_0, h) = \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

Then from the definition of a first partial derivative, we have

$$\lim_{h=0} F(x_0, y_0, h) = f'_x(x_0, y_0).$$

The oscillation $\omega(F)$ is defined as in functions of one variable as

$$\omega(F) = \omega(\delta) = M(F) - m(F),$$

where M is the maximum value and m is the minimum value of $F(x_0, y_0, h)$ in the interval $\delta(x,y)$. For continuous functions, we have

$$\lim_{\delta=0} \omega(\delta) = 0.$$

Baire proved (2) that $\delta(x,y)$ is an upper semi-continuous function in the neighborhood of the point (x_0, y_0), so that

$$\delta(x,y) < \delta(x_0, y_0) + \varepsilon. \qquad (1)$$

---

(1) E. J. Townsend. *Lectures on Theory of Functions of Real Variables.*
(2) Baire. *Lecons.* Art. 46.

Let (x_0, y_0) be any point in the given region R in which our assumed conditions hold. We can take a square s_1 containing the point (x_0, y_0) such that the inequality (1) holds. Any other point in the square may be made the center of an interval $[x-\delta(x,y),\ x+\delta(x,y)]$ such that the oscillation is less than an arbitrarily small constant η, for every $\delta(x,y) > 0$. s_1 may be made arbitrarily small. Divide s_1 into four equal parts by means of two lines parallel to the axes of ordinates and intersecting in the center of the square s_1. In one of these parts, say s_2, $\delta(x,y)$ takes on its lower limit. Continuing this process indefinitely, we have a sequence of squares each contained in the preceding and such that the limit of the area of the squares is zero. This sequence therefore defines a point (x_1, y_1) such that $\delta(x,y)$ has its lower limit at that point. But $\delta(x_1, y_1)$ is greater than zero and hence the lower limit is greater than zero. It is evident that the totality of points (x_1, y_1) is everywhere dense.

We wish now to show that the point (x_1, y_1) is a point of total differentiability. Since $f(x,y)$ is continuous in y,

$$F(x,y,h) = \frac{f(x+h,y) - f(x,y)}{h}$$

is also continuous in y for h fixed and different from zero and x and y lying in the given region R. Therefore we may write

$$|F(x_1, y_1+k, h) - F(x_1, y_1, h)| < \eta, \quad |k| < \beta, \ h \text{ fixed in } [x_1+\delta_1, x_1-\delta_1] \tag{1}$$

Also since $f'_x(x,y)$ exists at (x_1, y_1), we may write

$$|F(x_1, y_1, h) - F(x_1, y_1)| \equiv |F(x_1, y_1, h) - f'_x(x_1, y_1)| < \eta, \quad |h| < \delta_1 < \delta(x,y) \tag{2}$$

Combining (1) and (2), we have

$$|F(x_1, y_1+k, h) - f'_x(x_1, y_1)| < 2\eta, \quad |h| < \delta_1, |k| < \beta. \tag{3}$$

Since $f'_y(x_1, y_1)$ also exists, we may write

$$|f(x_1, y_1+k) - f(x_1, y_1) - k \cdot f'_y(x_1, y_1)| < \eta, \quad (x_1, y_1) \text{ in } R, |k| < \beta \tag{4}$$

From (3), we have

$$|f(x_1+h, y_1+k) - f(x_1, y_1+k) - h \cdot f'_x(x_1, y_1)| < 2\eta. \tag{5}$$

Combining (4) and (5), we obtain the condition that the total derivative exist at the point (x_1, y_1), namely

$$|f(x_1+h, y_1+k) - f(x_1, y_1) - [h \cdot f'_x(x_1, y_1) + k \cdot f'_y(x_1, y_1)]| < 3\eta, \quad (x_1, y_1) \text{ in } R$$
$$|h| < \delta_1; \ |k| < \beta.$$

In proving the second and third forms of the Law of the Mean for functions of two variables, we assumed that the first partial derivatives $f'_x(x,y)$ and $f'_y(x,y)$ are continuous in x alone and in y alone. Under these assumptions, we can prove the following theorem: (1)

THEOREM V. *The function $f(x,y)$, defined as in the second and third forms of the Law of the Mean, is totally differentiable in all points of the given region R.*

We proved in Section 10 that under the assumed conditions $f(x,y)$ is continuous in the two variables, x and y, together. Since $f'_x(x,y)$ exists, we have

$$\left|\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}-f'_x(x_0,y_0)\right|<\eta. \qquad (1)$$

Since $f'_x(x,y)$ is continuous in x and in y, we may write

$$\left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h}-\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right|<\eta. \qquad (2)$$

Combining (1) and (2), we have

$$\left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h}-f'_x(x_0,y_0)\right|<2\eta. \qquad (3)$$

From the existence of f'_y, we have

$$\left|\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}-f'_y(x_0,y_0)\right|<\eta. \qquad (4)$$

Combining (4) with (3), we have after multiplying them respectively by h and k,

$$\left|f(x_0+h,y_0+k)-f(x_0,y_0+k)+f(x_0,y_0+k)-f(x_0,y_0)-h\cdot f'_x(x_0,y_0)-k\cdot f'_y(x_0,y_0)\right|<3\eta,$$

or

$$\left|f(x_0+h,y_0+k)-f(x_0,y_0)-h\cdot f'_x(x_0,y_0)-k\cdot f'_y(x_0,y_0)\right|<3\eta,$$

which is the condition that the function $f(x,y)$ is totally differentiable at the point (x_0,y_0). Since this point is any point in the given region, the function $f(x,y)$ is totally differentiable at every point of the region.

The Law of the Mean as we have given it in the second form admits a simple geometrical interpretation. From the foregoing discussion, we know that $f(x,y)$ is totally differentiable at every point in the given region. We may therefore write the equation of

---

(1) E. J. Townsend. *Lectures on Theory of Functions of Real Variables.*

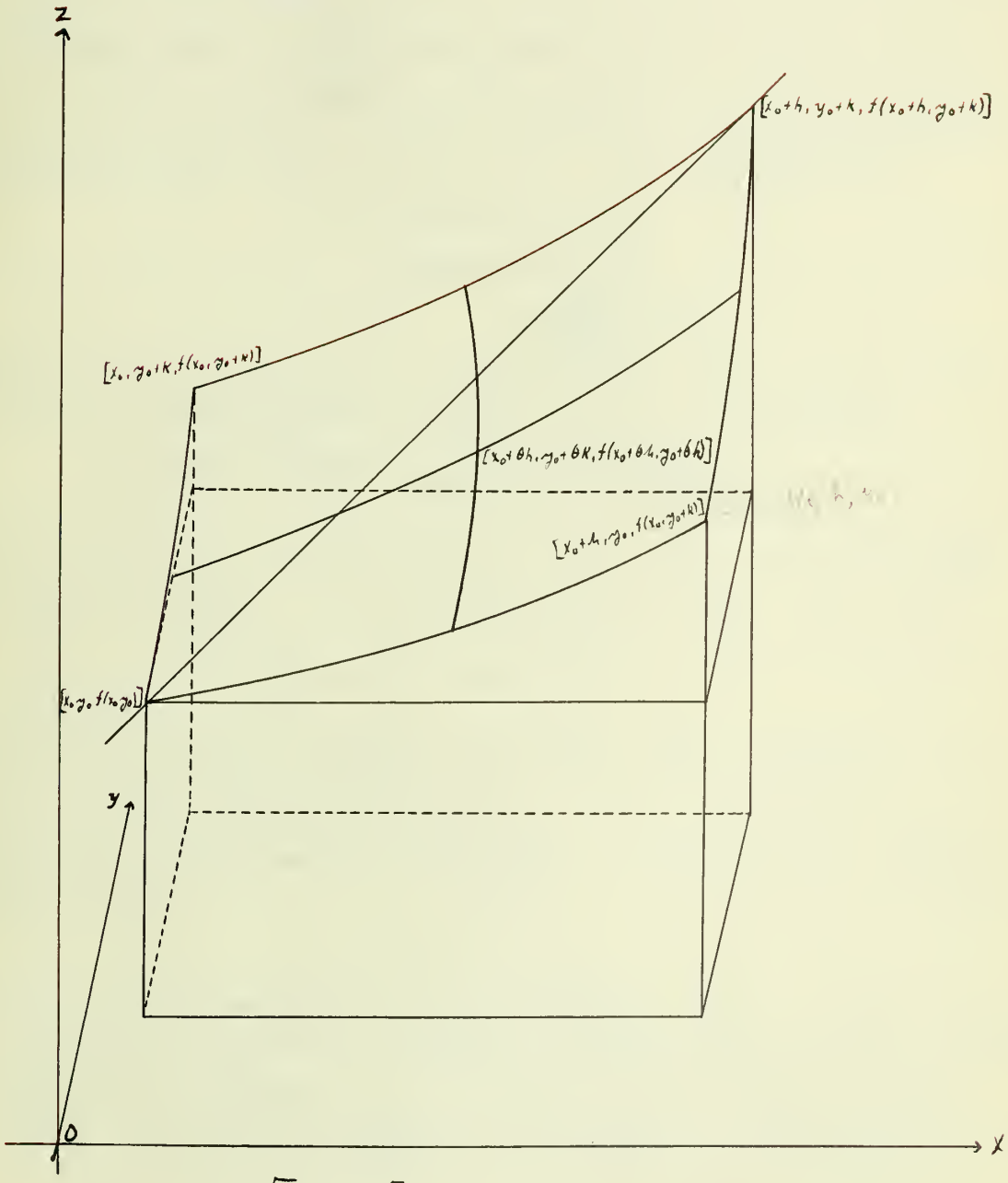
z
$[x_0+h, y_0+k, f(x_0+h, y_0+k)]$
$[x_0, y_0+k, f(x_0, y_0+k)]$
$[x_0+\theta h, y_0+\theta k, f(x_0+\theta h, y_0+\theta k)]$
$[x_0+h, y_0, f(x_0, y_0+k)]$
$[x_0, y_0, f(x_0, y_0)]$
y
0
x

Fig. 5.

the plane tangent to the surface $z = f(x,y)$ at the point $(x',y',z') \equiv [x_0+\theta h, y_0+\theta k, f(x_0+\theta h, y_0+\theta k)]$. This equation we obtain from the form: (1)

$$z - z' = \frac{\partial z}{\partial y'}(y - y') + \frac{\partial z}{\partial x'}(x - x').$$

The equation of the tangent plane at the given point is then

$$z-f(x_0+\theta h, y_0+\theta k)=f'_y(x_0+\theta h, y_0+\theta k)[y-(y_0+\theta k)]+f'_x(x_0+\theta h, y_0+\theta k)[x-(x_0+\theta h)]$$

which reduces to the form

$$x\cdot f'_x(x_0+\theta h, y_0+\theta k) + y\cdot f'_y(x_0+\theta h, y_0+\theta k) - z$$
$$= (x_0+\theta h)\cdot f'_x(x_0+\theta h, y_0+\theta k)+(y_0+\theta k)\cdot f'_y(x_0+\theta h, y_0+\theta k)+f(x_0+\theta h, y_0+\theta k).$$

The coefficients A, B, and C of x, y, and z are respectively

$$A = f'_x(x_0+\theta h, y_0+\theta k),$$
$$B = f'_y(x_0+\theta h, y_0+\theta k),$$
$$C = -1.$$

The equation of the line between the two points $[x_0, y_0, f(x_0, y_0)]$ and $[x_0+h, y_0+k, f(x_0+h, y_0+k)]$ may be written in the form:

$$\frac{x - x_0}{h} = \frac{y - y_0}{k} = \frac{z - f(x_0, y_0)}{f(x_0+h, y_0+k) - f(x_0, y_0)}.$$

The conditions that the plane $Ax + By + Cz = 0$ and the line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ be parallel are that $Al + Bm + Cn = 0$. (2)

We have that condition satisfied here. For,

$$Al + Bm + Cn$$
$$= h\cdot f'_x(x_0+\theta h, y_0+\theta k) + k\cdot f'_y(x_0+\theta h, y_0+\theta k) - [f(x_0+h, y_0+k) - f(x_0, y_0)]$$

But from the Law of the Mean we have

$$h\cdot f'_x(x_0+\theta h, y_0+\theta k) + k\cdot f'_y(x_0+\theta h, y_0+\theta k) = f(x_0+h, y_0+k) - f(x_0, y_0).$$

Therefore we have

$$Al + Bm + Cn = 0$$

which satisfies the condition for parallelism. That is, the tangent plane is parallel to the line joining the two points of the surface. We may say, then, that one of the planes of the sheaf of planes, intersecting in the line joining the two points of the surface, is parallel to the tangent plane at a mean point of the surface.

---

(1) Townsend and Goodenough. *First Course in Calculus.* p. 282
(2) Salmon. *Analytic Geometry of Three Dimensions.* Vol. I, p. 34.

13. HEDRICK'S THEOREMS EXTENDED TO FUNCTIONS OF TWO VARIABLES.

THEOREM VI. *A sufficient condition that the total differential of a function $f(x,y)$ be continuous, is that $\xi_1(h,k)$ and $\xi_2(h,k)$ occuring in the second form of the Law of the Mean shall be continuous in (h,k).*

In the second form of the Law of the Mean for functions of two variables, we replace θh by ξ_1 and θk by ξ_2, where it is evident that ξ_1 and ξ_1 are functions of (h,k). The Law of the Mean may then be written

$$f(x_0+h, y_0+k) - f(x_0, y_0) = h\cdot f'_x(x_0+\xi_1, y_0+\xi_2) + k\cdot f'_y(x_0+\xi_1, y_0+\xi_2). \quad (1)$$

Also it is evident that

$$|\xi_1| < |h|; \quad |\xi_2| < |k|.$$

Our assumptions in proving the Law of the Mean in this form led to the fact that $f(x,y)$ is totally differentiable in every point of the given region.

By hypothesis, we have

$$|\xi_1(h,k) - \xi_1(h_0,k_0)| < \varepsilon_1, \quad |h-h_0| < \rho_1, |k-k_0| < \rho_2$$
$$|\xi_2(h,k) - \xi_2(h_0,k_0)| < \varepsilon_2.$$

From the Law of the Mean, since

$$|\xi_1(h,k)| < |h|, \text{ and } |\xi_2(h,k)| < |k|,$$

and since ξ_1 and ξ_2 are assumed continuous, we have

$$\lim_{\substack{h\doteq 0\\ k\doteq 0}} \xi_1(h,k) = 0; \text{ and } \lim_{\substack{h\doteq 0\\ k\doteq 0}} \xi_2(h,k) = 0.$$

Since $f(x,y)$ is totally differentiable, we have

$$|[f(x_0+h, y_0+k) - f(x_0, y_0)] - h\cdot f'_x(x_0, y_0) + k\cdot f'_y(x_0, y_0)]| < \eta. \quad (2)$$

From the Law of the Mean, we may substitute for the first quantity in brackets its value given in (1) and obtain

$$|[h\cdot f'_x(x_0+\xi_1, y_0+\xi_2)+k\cdot f'_y(x_0+\xi_1, y_0+\xi_2)]-[h\cdot f'_x(x_0, y_0)+k\cdot f'_y(x_0, y_0)]| < \eta. \quad (3)$$

But since the total differential at the point (x,y) is

$$df(x,y) = h\cdot f'_x(x,y) + k\cdot f'_y(x,y),$$

the inequality (3) may be written

$$|df(x_0+\xi_1, y_0+\xi_2) - df(x_0, y_0)| < \eta, \quad |\xi_1| < |h|; \quad |\xi_2| < |k|.$$

This is the condition that the total differential shall be continuous in (x,y) at the point (x_0, y_0), which we were to show.

To show that the condition given in Theorem VI is not necessary, we give the following example of a function whose total differential is continuous and yet the functions $\xi_1(h,k)$ and $\xi_2(h,k)$ do not take on values which are everywhere dense and have the cardinal number of of continuum.

$$f(x,y) = x^3y^3\left(1 + \sin\frac{1}{xy}\right), \qquad (x\neq 0, y\neq 0),$$

$$= 0, \qquad (x=0,y=0),(x=0,y\neq 0),(x\neq 0,y=0).$$

This function is continuous in x alone and in y alone and in both variables together. Its first partial derivatives, and hence the total derivative, are continuous in each variable alone and in the two variables together.

$$f'_x(x,y) = 3x^2y^3\left(1 + \sin\frac{1}{xy}\right) - xy^2\cos\frac{1}{xy}, \quad (x\neq 0, y\neq 0)$$

$$= 0, \qquad (x=0,y=0),(x=0,y\neq 0),(x\neq 0,y=0).$$

$$f'_y(x,y) = 3x^3y^2\left(1 + \sin\frac{1}{xy}\right) - x^2y\cos\frac{1}{xy}, \quad (x\neq 0, y\neq 0,$$

$$= 0, \qquad (x=0,y=0),(x=0,y\neq 0),(x\neq 0,y=0).$$

The total differential $df(x,y)$ is zero for

$$\sin\frac{1}{xy} = -1;$$

that is, for a locus of points satisfying the condition

$$xy = \frac{2}{(4n-1)\pi}.$$

$$f_{xx}(x,y) = 6xy^3\left(1 + \sin\frac{1}{xy}\right) - \frac{y}{x}\sin\frac{1}{xy} - 4y^2\cos\frac{1}{xy}.$$

$$f_{yy}(x,y) = 6x^3y\left(1 + \sin\frac{1}{xy}\right) - \frac{x}{y}\sin\frac{1}{xy} - 4x^2\cos\frac{1}{xy}.$$

f_{xx} and f_{yy} are both positive for the locus of points $xy = \frac{2}{(4n-1)\pi}$, since at those points $\sin\frac{1}{xy}$ and $\cos\frac{1}{xy}$ are negative one and zero respectively. Hence $f(x,y)$ must ne at a minimum at those points. The same is true of the locus of points $xy = \frac{2}{(4n+3)\pi}$. Since $f(x,y)$ is continuous and not constant, there must be some points lying between these two loci at which both the first partial derivatives are negative and hence at which the total differential is negative. But from the Law of the Mean, we have

$$f(x_0+h, y_0+k) - f(x_0,y_0) = df(x_0+\xi_1, y_0+\xi_2).$$

The left-hand member of this equation is always positive in the region of the locus $xy = 1/(4n-1)\pi$ since the function is at a minimum at the

locus. At some points, say (x_1, y_1), the right-hand member is negative in the region of the locus. Hence ξ_1 and ξ_2 can never be located at any of the points (x_1, y_1). These points are all points in a region and hence the points which ξ_1 and ξ_2 do not take on have the cardinal number of the continuum.

Another example showing the condition of Theorem VI not a necessary condition is

$$f(x,y) = (x+y)^3\left[1 + \sin\frac{1}{x+y}\right], \quad (x\neq 0, y\neq 0),$$

$$= 0, \qquad (x=0, y=0), (x=0, y\neq 0), (x\neq 0, y=0).$$

The locus of points in this case for which $f(x,y)$ is a minimum is

$$x + y = \frac{2}{(4n-1)\pi}.$$

THEOREM VII. *The functions $\xi_1(h,k)$ and $\xi_2(h,k)$ defined by the equation*

$$f(x_0+h, y_0+k) - f(x_0, y_0) = h\cdot f'_x[x_0+\xi_1(h,k), y_0+\xi_2(h,k)] + k\cdot f'_y[x_0+\xi_1(h,k), y_0+\xi_2(h,k)]$$

are continuous in (h,k) in the closed region $(0 \leq h \leq a_1 - a_0;$ $0 \leq k \leq b_1 - b_0)$ if $\xi_1(h,k)$ and $\xi_2(h,k)$ are single-valued functions of (h,k) and conversely if $h(\xi_1, \xi_2)$ and $k(\xi_1, \xi_2)$ are single-valued functions of (ξ_1, ξ_2).

The proof of this theorem is similar to the proof of Theorem XII in Chapter I.

UNIVERSITY OF ILLINOIS-URBANA
3 0112 086832372